$\mathrm{Ind}_a = \mathrm{Ind}_t$

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Honour School of Mathematics Part C

Trinity 2019

Contents

Introduction

Theorem 0.1 *Let D be an elliptic partial differential operator of order one*[1] *on a smooth, closed, even dimensional manifold M. Denote by $\sigma_D \in K(T^*M)$ its symbol class. Then*

$$\operatorname{Ind}(D) = \int_{T^*M} \operatorname{ch}(\sigma_D) \operatorname{Todd}(TM \otimes \mathbb{C})$$

This paper is a short exposition to two modern approaches of the Atiyah-Singer Index Theorem utilizing methods of operator theory.

The first three chapters set up the stage. In Chapter 1, we explain standard properties of EPDOs and end with how we can apply functional calculus. In Chapter 2 we prove Bott periodicity for stable C^* homology theories. In Chapter 3, we explain topological background: the construction of topological index via orientations and the Chern character map. In Chapter 4, we reformulate Thm. 0.1 as

$$\operatorname{Ind}_a = \operatorname{Ind}_t$$

In the last two chapters we look into two proofs. In Chapter 5, we follow Higson's asymptotic K-theory $*$-morphism proof for EPDOs of order one, [Hig93]. This notion replaces the use of pseudodifferential calculus. In Chapter 6 we look at Debord and Lescure's, [DL08], which utilizes KK-theory.

The common intersection between the two proofs is the notion of deformation groupoids. Groupoid algebras have both a geometrical and analytic flavour, enabling analysis of singular behaviors. The tools presented leads nicely to the index proof for foliations, [CS84]. The aim is therefore to give an introduction to the noncommutative language for index theory.

We will closely follow the notes [Hig14],[Ebe14], and [DL08]. Conventions and prerequisites are stated in the start of each chapter. It is recommended that the reader has background in the basics of C^* algebras, such as the Gelfand transform, GNS construction, and Hilbert Modules. It would be helpful to keep [Bla06] by hand.

[1] This is the essential case. Higher order cases reduces to the order one case, which classically requires pseudodifferential operators.

Chapter 1
Elliptic Partial Differential Operators

The prerequisites are elementary Fourier theory[Ray91, Ch.1] and some basic theory of unbounded linear operators, [Tao09], [Wil14, Ch. 7]. For $s \in \mathbb{R}$, W^s denotes the completion of $\mathcal{S}(\mathbb{R}^n)$, the *Schwartz functions* on $\mathbb{R}^n$, by the *Sobolev norm*

$$||f||_s^2 := \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi.$$

All structures considered are smooth. $\Gamma(M; E)$ denotes smooth sections of a bundle $E \to M$. We will be focusing on partial differential opeartors of order 1, but will state otherwise when working in generality.

In the first section, we define EPDOs, and explain how we may pass classical inequalities to that on Hermitian vector bundles over closed manifolds. We prove C^∞ Hodge Theorem, Thm. 1.12.

In the second section, we give a geeneral construction of EPDOs. In the last section, we introduction the language of functional calculus, which will be the basis of constructing K-theory classes in Chapter 4. This section can be omitted in on first read and reviewed again.

1.1 Partial Differential Operators

Definition 1.1 Let M be a smooth manifold and $E_i \to M$ be two smooth vector bundles over $\mathbb{R}$ (or $\mathbb{C}$). A *partial differential operator* (PDO) $P : \Gamma(M, E_0) \to \Gamma(M, E_1)$ *of order* k is a linear map such that:

1. P is local: if $s \in \Gamma(M, E_0)$, $s|_U = 0$ for some open subset $U \subseteq M$, then $(Ps)|_U = 0$.

2. If $x : U \to \mathbb{R}^n$ is a chart, $\phi_i : E_i\Big|_U \to U \times \mathbb{R}^{p_i}$ a trivialization, then the localizaed operator $\phi_1 \circ P \circ \phi_0^{-1}$ can be expressed as

$$(\phi_1 \circ P \circ \phi_0^{-1})(f)(y) = \sum_{|\alpha| \leq k} A^{(\alpha)}(y) \frac{\partial^\alpha}{\partial x_\alpha} f(y)$$

   for each $f \in C^\infty(U, \mathbb{R}^{p_0})$ where $A^\alpha : U \to M_{p_1,p_0}(\mathbb{R})$ is smooth.

1 is necessary for 2 to make sense. We now give a pointwise definition of symbol.

Definition 1.2 Let $P : \Gamma(M, E_0) \to \Gamma(M, E_1)$ be a PDO of order k. Let $y \in M$, $\xi \in T_y^*M$ and $e \in (E_0)_y$. *Pick* $f \in C^\infty(M; \mathbb{R})$ with $f(y) = 0$ and $(df)_y = \xi$; $s \in \Gamma(M, E_0)$ with $s(y) = e$. We define the *symbol* at y, ξ of P to be the linear map

$$\sigma_P(y, \xi) : (E_0)_y \to (E_1)_y$$

$$\sigma_P(y, \xi)(e) := \frac{i^k}{k!} P(f^k s)(y) \in (E_1)_y$$

Lemma 1.3 *The expression* $\mathrm{smb}_k(P)(y, \xi)(e)$ *only depends on* y, ξ, e.

Proof: In local coordinates, choose a chart U, for which both E_0 and E_1 are trivial, and $\xi = \sum \xi_i dx^i$. Then

$$\frac{i^k}{k!} P(f^k s)(y) = \frac{i^k}{k!} \sum_{|\alpha| \le k} \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} A^\alpha(y) \frac{\partial^{|\beta|}}{\partial x_\beta}(f^k)(x) \frac{\partial^{|\gamma|}}{\partial x_\gamma}(s)(x)$$

So by induction,

$$\mathrm{smb}_k(P)(y, \xi)(e) = i^k \sum_{|\alpha| = k} A^\alpha(y) \xi^\alpha e$$

□

If $P : \Gamma(M, E_0) \to \Gamma(M, E_1)$ is a PDO of order 1, $s \in \Gamma(M, E_0)$ the symbol can be computed $\sigma_P(x, df_x)s = i[P, f]s$. Hence,

Definition 1.4 Let D be an order one PDO on $\Gamma(M, E)$. The *symbol* of D is a bundle morphism,

$$\sigma_D : T^*M \to \mathrm{End}(E), \quad \sigma_D : df \mapsto i[D, f]$$

Sobolev Spaces on Manifolds

Definition 1.5 Pick a finite cover of a closed n manifold M , $\{h_i : U_i \xrightarrow{\simeq} \mathbb{R}^n\}$, trivializations $\{\phi_i : E|_{U_i} \xrightarrow{\simeq} U_i \times \mathbb{R}^n\}$ and a partition of unity μ_i subordinate to $\{U_i\}$. The Sobolev norm of $u \in \Gamma(M, E)$ is

$$||u||_{k,M}^2 := \sum_i ||\mu_i \phi_i \circ u \circ (h_i)^{-1}||_k^2$$

$W^s(M; E)$ is the completion of $\Gamma(M; E)$ with respect to this norm. The key lemma in showing independence of covering is

Lemma 1.6 *Let* $\phi : U' \to V'$ *be a diffeomoprhism of open subsets of* $\mathbb{R}^n$ *with* $U \subseteq U'$ *and* $V = \phi(U) \subseteq V'$ *be relatively compact. Then for all* $s \in \mathbb{Z}$, $u \mapsto u \circ \phi$ *extends to a bounded map*

$$W^s(V) \to W^s(U)$$

$W^s(U)$ *denotes closure of* $C_c^\infty(U) \subseteq \mathcal{S}(\mathbb{R}^n)$ *under the Sobolev norm.*

Proof: Assume $s = k \in \mathbb{N}$ and $u \in C_c^\infty(V)$. We use equivalent norm

$$||u \circ \phi||_k^2 = \sum_{|\alpha| \le k} \int |D^\alpha (u \circ \phi)|^2 \, dx.$$

Because U and V are relatively compact, by change of variables formula,

$$\int |D^\alpha(u \circ \phi)(x)|^2\,dx \le C(\phi,\alpha) \sum_{|\beta|\le|\alpha|} \int |(D^\beta u)\circ\phi|^2\,x$$
$$= \int C \sum_{|\beta|\le|\alpha|} \int |(D^\beta u)|^2 |\det D\phi|^{-2}\,dy \le C'||u||_k^2$$

For $-k < 0$, we apply duality of Sobolev spaces. □

Corollary 1.7 *The equivalence class of norm $||\cdot||_k$ does not depend on the choices in Def. 1.5. Further, if u has compact support in a coordinate neighborhood U_i, then $||\phi_i \circ u \circ h_i^{-1}||_{k,\mathbb{R}^n} \le C||u||_{k,M}$ where C depends on the trivializations and charts.*

Inequalities on Sobolev Manifolds

We utilize a small lemma, [Ebe14, Lem. 3.3.6], to pass standard inequalities in PDE theory to manifolds.

Lemma 1.8 *(Peter and Paul estimate in $\mathbb{R}^n$) Let $r < s < t \in \mathbb{R}$. Then for each $\varepsilon > 0$, there exists a $C(\varepsilon) > 0$ such that for all $u \in \mathcal{S}$, the Schwartz space of $\mathbb{R}^n$, the estimate,*

$$||u||_s \le \varepsilon||u|_t + C(\varepsilon)||u||_r$$

holds.

Theorem 1.9 *Let M be a closed manifold and $E \to M$ a Hermitian vector bundle. Let $P : \Gamma(M;E_0) \to \Gamma(M;E_1)$ be a PDO of order k. Then:*

1. *The inclusion $W^l \to W^k$ is injective for $k > l$.*
2. *(Sobolev embedding) For $l > \frac{n}{2} + k$, the elements of W^l are C^k sections.*[1]
3. *(Rellich compcatness) The inclusion $W^l \to W^k$ is compact if $l > k$.*
4. *(Gardings inequality) If P is elliptic, then there is a constant C such that*

   $$||u||_{k+l,M} \le C(||u||_{l,M} + ||Pu||_{l,M})$$

   for all $u \in W^{k+l}$.
5. *P induces a continuous operator $W^{l+k} \to W^l$ for all l.*
6. *(Duality) The map $W^k \to (W^{-k})^*$ given by $u \mapsto \langle v, \langle u,v\rangle\rangle$ is an antilinear isomorphism.*
7. *(Elliptic regularity) Let $Pu = f$ (takes place in W^{r-k}) $f \in W^l$, $u \in W^r$ for some integer r, then $u \in W^{l+k}$.*

[1] k times differentiable.

Proof: The translation proofs are similar. We illustrate 4. Let $u_i := \phi_i \circ u \circ h_i^{-1}$. Then

$$||u||_{k+l,M} \leq C \sum_i C_i(||\mu_i u_i||_{l,\mathbb{R}^n} + ||P\mu_i u_i||_{l,\mathbb{R}^n})$$

by Garding's in $\mathbb{R}^n$. We analyze the last term. If $a_i \in C_c(U_i)$, with $a_i\mu_i = \mu_i$, then

$$||P\mu_i u_i||_{l,\mathbb{R}^n} \leq C_i\left(||[P,\mu_i]a_i u_i||_{l,\mathbb{R}^n} + ||\mu_i P a_i u_i||_{l,\mathbb{R}^n}\right) \leq C_i\left(||[P,\mu_i]a_i u_i||_{l,\mathbb{R}^n} + ||Pu||_{l,\mathbb{R}^n}\right)$$

For each $\varepsilon > 0$, the first term is bounded by $\varepsilon||a_i u||_{k+l} + C(\varepsilon)||a_i u||_l$, using Lem. 1.8. We bound again by Cor. 1.7, to norm on manifold. Result follows by letting $\varepsilon \to 0$. □

Garding's and Rellich implies that if P is an EPDO of order k, then $P : W^{k+l} \to W^l$ has finite dimensional kernel and closed image. Regularity and Sobolev embedding implies

Theorem 1.10 *(local regularity) If P is an EPDO, $P : W^{k+l} \to W^l$, with $Pu = f$ and f is smooth section, then u is smooth.*

Hence, dimension of kernel does not depend on l. We now consider the formal adjoint. [2]

Definition 1.11 Given Riemannian closed manifold M, with Hermitian bundles $E_i \to M$, P a PDO of order k, a *formal adjoint* $P^* : \Gamma(M; E_1) \to \Gamma(M; E_0)$ is a PDO such that $\langle Px_1, x_2\rangle = \langle x_1, P^*x_2\rangle$ for all $x_i \in \Gamma(M; E_i)$. [3]

If P is an EPDO of order k, P^* exists, is unique and is also an EPDO of order k. Further, its symbol can be computed pointwise. Hence, the Fredholm index is independent of l. For the $l = 0$ in Thm. 1.9, the induced map of P, $Q : W^k \to L^2$, gives

$$L^2(M; E_1) = \operatorname{im} Q \oplus \ker Q^*.$$

By Thm. 1.10, $\ker Q^* = \ker P^*$, giving

Theorem 1.12 *(C^∞ Hodge theorem) Let $P : \Gamma(M; E_0) \to \Gamma(M; E_1)$ be an EPDO of order k on a closed manifold M, then*

$$\Gamma(M; E_1) = \operatorname{im} P \oplus \ker P^*$$

and P is a Fredholm operator.

1.2 Elliptic Complexes and Hodge Decomposition

We recite the construction elliptic operators due to Atiyah, [AS68].

Definition 1.13 Let M be a smooth manifold. An *elliptic complex* $\mathcal{E} = (E_*, P)$ of length n is a sequence

$$0 \to \Gamma(M, E_0) \xrightarrow{P_1} \Gamma(M, E_1) \xrightarrow{P_2} \cdots \xrightarrow{P_n} \Gamma(M, E_n) \to 0$$

of differential operators of order 1 between complex vector bundles[4] such that

[2] There is a more general definition. We specialize to our context.
[3] This definition holds more generally for M noncompact.
[4] We have restricted to P_i of order 1, sufficient for our purpose.

1. $P_i \circ P_{i-1} = 0$ and
2. for each nonzero cotangent vector $\xi \in T^*_m M$ the sequence

$$0 \to (E_0)_m \xrightarrow{\sigma_{P_1}(\xi)} (E_1)_m \xrightarrow{\sigma_{P_2}(\xi)} \cdots \xrightarrow{\sigma_{P_n}(\xi)} (E_n)_m \to 0$$

is exact.

We give an example.

Lemma 1.14 *Let V be a finite dimensional vector of dimension n. For $\xi \in V^*$, Define $\epsilon_\xi : \Lambda^p V^* \to \Lambda^{p+1}V^*, w \mapsto \xi \wedge w$. For $\xi \neq 0$, we have exact complex,*

$$0 \to \Lambda^0 V^* \xrightarrow{\epsilon_\xi} \Lambda^1 V^* \xrightarrow{\epsilon_\xi}\to \cdots \xrightarrow{\epsilon_\xi} \Lambda^n V^* \to 0.$$

Proof: Consider adjoint, $i_v : \Lambda^p V^* \to \Lambda^{p-1}V^*, \quad i_v(w)(v_2, \ldots, v_p) = w(v_1, \ldots, v_p)$. Hence, for $\xi \in V^*, v \in V$, $\epsilon_\xi i_v + i_v \epsilon_\xi = \xi(v)$, acts as multiplication by $\xi(v)$. If $\xi \wedge w = 0$, choose $v \in V$ such that $\xi(v) = 1$, giving $w = \xi(v)w = \epsilon_x(i_v w)$. Exactness follows. □

Proposition 1.15 *The de Rham complex of a manifold M of dimension n is $(d_p : \Omega^p(M) \to \Omega^{p+1}(M))_{0\leq p \leq n}$, where $\Omega^p(M) = \Gamma(M, \Lambda^p T^* M))$ This is an elliptic complex of length n.*

Proof: $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is a PDO of order 1. By Def. 1.4, for $f \in C^\infty(M)$.

$$\sigma_d(df)w = i(d(fw) - f dw) = i(df \wedge w)$$

Thus, fiberwise, $\xi \mapsto i\xi \wedge w$ for $\xi \in T^*_m M$, $w \in \Lambda^p T^*_m M$, we are reduced to Lem. 1.14. □

Definition 1.16 Let (E_*, P) be an elliptic complex over a Riemannian manifold, where each bundle $E_i \to M$ has Hermitian bundle metric. There is a bundle metric on $\oplus_i E_i$. Direct sum gives an operator

$$P : \bigoplus_i \Gamma(M, E_i) \to \bigoplus_i \Gamma(M, E_i).$$

Lemma 1.17 *Let $0 \to V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2}\to \cdots V_n \to 0$ be a cochain of complex of finite dimensional hermitian vector spaces. The following are equivalent.*

1. *The complex is exact.*
2. *The linear map $f + f^* : V_* \to V_*$ is exact.*

Proof: Suppose $f+f^*$ is an isomoprhism. f^* defines a chain homotopy, from 0 to $f^*f + ff^* = (f+f^*)^2$ giving 0 map in cohomology. The converse is element chase. □

Corollary 1.18 *Let (E_*, P) be an elliptic complex, and $E = \bigoplus_{i\geq 0} E_i$. Let P^* be adjoint of P. Then $P + P^*$ on $\Gamma(M; E)$ is elliptic. Hence, the restricted operators,*

$$P + P^* : \Gamma(M, E^\pm) \to \Gamma(M, E^\mp)$$
$$E^+ := \bigoplus_i E_{2i}, E^- := \bigoplus_i E_{2i+1}$$

are elliptic (formally self adjoint too).

Proof: This follows from Lem. 1.17 and that symbol of adjoint can be computed pointwise. □

From the elliptic complex $\mathcal{E} = (E_*, P)$ on closed manifold M, we define its cohomology [5]

$$H^p(\mathcal{E}) := \frac{\ker(P_p : \Gamma(M, E_p) \to \Gamma(M, E_{p+1})}{\operatorname{im}(P_{p-1} : \Gamma(M, E_{p-1}) \to \Gamma(M, E_p)}$$

$$D = P + P^*, \quad \Delta = D^2, \quad \mathcal{H}^p(\mathcal{E}) := \ker \Delta$$

Directly applying Thm. 1.12, we have

Theorem 1.19 *(The Hodge decomposition theorem) Let M be a closed Riemannian manifold and $\mathcal{E}$ an elliptic complex on M. Then there are orthogonal decompositions:*

1. $\Gamma(M; E) = \ker(D) \oplus \operatorname{im}(D)$.
2. $\ker(P) = \operatorname{im}(P) \oplus \ker(\Delta)$.
3. *The natural map* $\ker(\Delta) \to H(\mathcal{E})$ *is an isomoprhism.*

We apply results to the de Rham complex, $(\Omega^*(M); d)$. By Cor. 1.18 we have $D = P + P^* : \Omega^+(M) \to \Omega^-(M)$. Hodge decomposition yields,

$$\operatorname{Ind}(D) = \sum_{p=0}^{n} (-1)^p \ker(\Delta) = \sum_{p=0}^{n} (-1)^p H^p(M)$$

representing the index of an operator as the Euler characteristic of M!

1.3 Functional Calculus on Symmetric EPDOs

In PDE theory, an operator $L : H \to H$, is often densely defined. Being *essentially self adjoint*, is a sufficient condition for nice spectral theory.

Theorem 1.20 *[Wil14, Thm. 7.34] (Spectral Theorem for Unbounded Self-Adjoint Operators) Suppose that T is an unbounded selfadjoint operator in H. Then there is a locally compact space Y, a Randon measure μ, a continuous function $h : Y \to \mathbb{R}$ and a unitary $U : H \to L^2(Y, \mu)$ such that*

1. $UD(T) = D(M_h) := \{f \in L^2(Y, \mu) \ : \ hf \in L^2(Y, \mu)\}$
2. $UTv = M_h Uv$ *ofr all* $v \in D(T)$

where M_h is multiplication operator.

As a corollary, we deduce, [Wil14, Cor. 7.3.5].

Theorem 1.21 *(Functional Calculus for Unbounded Self-Adjoint Operators) Suppose that T is a self-adjoint operator in H. Then there is a $*$-homomorphism Φ from the bounded Borel functions on $\mathbb{R}$, regarded as a C^* algebra with sup norm, into $B(H)$ such that*

$$\Phi(r_\pm) = R(\pm i, T), \quad r_\pm(x) := \frac{1}{\pm i - x}$$

where $R(j, T) := (T - jI)^{-1}$ for $j \in \mathbb{C}$. [6]

[5] noting that Δ maps $\Gamma(M, E_p)$ to itself.

[6] That $R(\pm i, T)^{-1}$ is bounded oeprator follows from self adjoint.

Proof: We may assume that T acts on $L^2(Y,\mu)$ by a continuous function $h: Y \to \mathbb{R}$. In Thm. 1.21, $R(i,T)$ is in fact given by M_g, with $g = \frac{1}{i-h}$. Let F be any bounded borel function on R, then $F \circ h$ is induces a bounded operator $M_{F\circ h}$ on $L^2(Y,\mu)$ with norm bounded by $||F \circ h||_\infty$. Define $*$-homomorphism $\Phi(F) = M_{F\circ h}$, which recovers

$$\Phi(r_+) = M_g = R(i,T), \quad \Phi(r_-) = \Phi(r_+)^* = R(-i,T)$$

□

To illustrate, let $D_+ = \partial_x + x, D_- = -\partial_x + x$.

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} : C_c(\mathbb{R})^2 \to L^2(\mathbb{R})^2$$

This induces bounded operator

$$\frac{1}{\sqrt{(1+D^2)}} D : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$$

Theorem 1.22 *If D is a symmetric PDO on a closed manifold[7], then D is essentially self-adjoint.*

Proof: If u is orthogonal to the range of either $D \pm iI$, then for all $w \in C_c^\infty(M;S)$

$$\langle (D \pm iI)w, u\rangle = 0$$

hence, u by definition lies in minimal domain[8] of $(D \mp iI)u = 0$, which lies in maximal domain, from elliptic regularity, Thm. 1.9. Then for a choice of smooth sections $u_n \to u$, $(D \mp iI)u_n \to 0$

$$0 = \lim_n ||(D \mp iI)u_n||^2 \geq \lim_n ||u_n||^2 = ||u||^2$$

□

Standard results for bounded compact self adjoint operators, [TZ96, Pg. 515], may be extended without much effort too.

Definition 1.23 Let D be an essentially self-adjoint operator on a Hilbert space H. We say that D has *compact resolvent* if for all $f \in C_0(\mathbb{R})$, the operator $f(\bar{D})$ defined by spectral theorem is compact. $\bar{D}$ denotes the closure of D.

With the above definition, and Hilbert space theory, [Wil14, Ch. 3],

Corollary 1.24 *If D is an essentially self adjoint operator on a Hilbert space H, and if D has compact resolvent, then the kernel of $\bar{D}$ is finite dimensional, range is closed, and $H = \ker(\bar{D}) \oplus \operatorname{im}(\bar{D})$.*

We return to order one symmetric PDOs.

Proposition 1.25 *Let M be a closed manifold. If D is an essentially self adjoint operator on $L^2(M;E)$ and if the domain of formal adjoint of D is $W^1(M;E)$, then D has compact resolvent.*

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[7]Note that symmetric PDOS ares closable operators.

[8]Minimal domain is the domain of the closure, and maximal domain is domain of adjoint.

Proof: By closed graph theorem, $(D \pm iI)^{-1} : L^2(M) \to H^1(M)$ is bounded. By Rellich lemma, the operators $(D \pm iI)^{-1}$ are compact. As $(x \pm i)^{-1}$ generate $C_0(\mathbb{R})$, approximation argument shows $f(D)$ is compact for all f. □

It suffices to consider when $\operatorname{dom} \bar{D}$ equal $W^1(M; E)$, but this is Thm. 1.9 restated.

Corollary 1.26 *Let D be a symmetric order one, EPDO on a closed manifold M, acting on a smooth vector bundle E. The domain of $\bar{D}$ is $W^1(M; E)$ and D has compact resolvent.*

Hence we may define maps as

$$\phi_D : C_0(\mathbb{R}) \to L^2(M; E), \quad , f \mapsto f(\bar{D}), \quad (x \pm iI)^{-1} \mapsto (\bar{D} \pm iI)^{-1}$$

This construction will be used repeatedly Chapter 4 to construct K-theory classes, via an identification. From now on, for symmetric EPDOs, we implicitly replace D with $\bar{D}$.

Chapter 2
Operator K Theory

The main prerequisites are basiccs of C^* algebra, polar decompositions, [Wil19, Ch. 1-5]. For the Morita invariance subsection, the constructions and definitions of Hilbert modules, [Bla06, Ch. 2.7], [JT91, Ch.1]. Given a C^* algebra A and a Hilbert module[1] $\mathcal{E}_A$, the sets $\mathrm{Mor}(\mathcal{E})$ and $\mathcal{K}_A(\mathcal{E})$ will denote the adjointable and compact morphisms respectively.

We review basic operator and topological K-theory [NdK17, Ch. 1-3], [Hat17, Ch. 1]. $\mathcal{K}$ will denote the compact operators on an infinite dimensional separable Hilbert space. A^+ will denote unitzation and $\otimes$ is used freely when the C^* algebras involved are nuclear C^* algebras, [Tak64], this includes commutative C^* algebras and $\mathcal{K}$. We work in the category $C^*\,\mathrm{Alg}$ with objects as separable C^* algebras, and morphisms as $*$-homomoprhisms.

The first section motivates how one pass from topological K-theory to operator K-theory. We discuss nice properties of operator K-theory and prove that operator K-theory satisfies the axioms for a stable C^* algebra homology theory. The section ends with Morita equivalences, the fundamental result in constructing new K-theory classes. In the second section, we prove Bott periodicity from the perspective of operator K-theory.

2.1 Extending Topological K-Theory

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let K_0 be *algebraic* K-theory group on rings R. The topological K-theory of $K^0_{\mathbb{F}}(X)$ of X is defined on the category of compact Hausdorff spaces. We recall how we can pass from K^0 to K_0 via Swan's theorem. Let $S = C(X)$, ring of continuous functions from X to $\mathbb{F}$. If $E \xrightarrow{p} X$ is $\mathbb{F}$-vector bundle over compact Hausdorff X, the map

$$E \mapsto \Gamma(X, E)$$

induces an isomorphism of categories from the category of locally trivial vector bundles over X to the category of finitely generated projective S-modules. Compactness is necessary for being finitely generated - which needs a finite partition of unity. [2]

Via algebraic K-theory, if X is a locally compact Hausdorff space denote

$$K^0_{\mathbb{F}}(X) := K_0(C_0(X))$$

Further, if X is a compact Hausdorff space and A is a closed subspace, there is an exact

[1] We take the convention of considering *right* module action.

[2] Projectivity follows from a choice of Hermitian metric inducing bundle E', such that $E \oplus E' \cong X \times \mathbb{F}^k$.

sequence induced by the inclusion[3] $A \hookrightarrow X$

$$K^0(X \setminus A) \to K^0(X) \to K^0(A)$$

from algebraic K-theory. The extension ends here. For higher topological K-theory we define

$$K_{\mathbb{F}}^{-n}(X) = K_{\mathbb{F}}(S^n X), \quad \text{for } n > 0$$

where S is the suspension functor. This forms generalized cohomology theories. The difference for $\mathbb{R}$ and $\mathbb{C}$ comes in $K_{\mathbb{C}}(X) \cong K_{\mathbb{C}}^{-2}(X)$ and $K_{\mathbb{R}}(X) \cong K_{\mathbb{R}}^{-8}(X)$.

Unfortunately, $K_1^{alg}(C_c(X))$ is distinct to $K_{\mathbb{C}}^{-1}(X)$. [4] *Operator K* theory, however, extends complex K theory well. We explore this.

Idempotents and Projections

We follow Nest's treatment, [NdK17]. Let A be a C^* algebra,

$$\mathrm{Idem}(A) := \{e \in A \,:\, e^2 = e\}$$

$$\mathrm{Proj}(A) = \{p \in \mathrm{Idem}(A)\,;\, p^* = p\}$$

If A is unital,

$$U(A) := \{u \in A \,:\, u^*u = uu^* = 1\}$$

all sets are given the subspace topology. Define,

$$V(A) := \pi_0(\mathrm{colim}\,\mathrm{Idem}\,M_n(A)) \simeq \mathrm{colim}_n\,\pi_0(\mathrm{Idem}\,M_n(A))$$

The triple $(V(A), +, [0])$ forms a commutative monoid. For representatives $a \in M_n(A), b \in M_m(A)$, $[a] + [b] = [a \oplus b]$, $a \oplus b \in M_{n+m}(A)$ is the block sum.

Definition 2.1 For unital C^* algebra A, $K_0(A) := G(V(A))$ where $G(-)$ is group completion.

We make in fact work with projections. The inclusion $i \,:\, \mathrm{Proj}(A) \hookrightarrow \mathrm{Idem}(A)$ is a homotopy equivalence, [NdK17, Lemma 2.12], inducing isomoprhism $\mathrm{colim}_n(\mathrm{Proj}\,M_n(A)) \to \mathrm{colim}\,\pi_0(\mathrm{Idem}\,M_n(A))$. Hence,

$$K_0(A) \simeq G\Bigg(\mathrm{colim}_n\,\pi_0(\mathrm{Proj}(M_n(A))\Bigg)$$

We do a computation.

Lemma 2.2 *Every $G \in GL_n(\mathbb{C})$ is path connected in $GL_n(\mathbb{C})$ to a unitary matrix.*

Proof: From Gramm-Schimdt, write $U = GS_1 \cdots S_n$, where S_i are elementary operators path connected in $GL_n(\mathbb{C})$ to the identity. $\square$

[3]The indexing of $\mathbb{F}$ is removed as proofs are identical.

[4] A comparison is in, [Ros05]. Rosenberg proved the two coincides for stable C^* algebras.

Lemma 2.3 *Let* $n \in \mathbb{N}$ *the trace map*

$$\mathrm{Tr} : \pi_0(\mathrm{Proj}(M_n(\mathbb{C})) \to \{0, 1, \ldots, n\}$$

is a bijection. So $K_0(\mathbb{C}) \simeq \mathbb{Z}$*.*

Proof: Tr of a projection coincides with its Jordan normal form - consisting of 0 and 1 in diagonal. By Lem. 2.2, there is a path of idempotents from projections to their normal form. By Lem. 2.10, the path can be given as projections; Tr is well defined and bijective. □

Murray-von Neumann Equivalence

The advantage of operator K-theory is the tractable description of its elements.

Definition 2.4 Let $p, q \in \mathcal{P}(A)$

1. *Homotopy equivalence* $p \sim_h q$*:*
2. *Unitary equivalent* $p \sim_u h$*:* if $upu^* = q$ for some unitary $u \in A$
3. *Murray-von Neumann equivalent* $p \sim_m h$*:* if there is a $v \in A$, $v^*v = p$ and $v^*v = q$.

In finite matrix algebras, 1 implies 2 implies 3. None is reversible. However, these notions coincide in the K_0 group of A. We set this up.

Proposition 2.5 *Suppose* $p, q \in \mathrm{Proj}(A)$ *such that* $p \sim_h q$ *by the path* p_t*, then there is a path of unitaries such* u_t *such that* $u_t^* p u_t = p_t$

Proof: By uniformity and compactness argument, we assume that $||p - p_t|| < 1$. There is a path from unitary $2p - 1$

$$x_t = pp_t + (p-1)(p_t - 1)$$

As $||x_t - 1|| \leq ||2p-1||\,||p_t - p|| < 1$, it is convertible for all $t \in [0, 1]$. Write $x_t = u_t|x_t|$. in its polar decomposition, then direct computation (in the given order) yields

$$px_t = pp_t = xp_t, \quad p_t x_t^* x_t = pp_t p = x_t^* x_t p_t$$

$$p_t^2 x_t^* x_t = x^* x_t p_t^2, \quad p_t |x_t| = |x_t| p_t$$

so that $u_t^* p u_t = p_t$. □

Corollary 2.6 *Suppose* $p, q \in \mathrm{Proj}(A)$ *with* $p \sim_h q$ *then* $p \sim_u q$*.*

Similarly, if $p \sim q$, then $p \sim_u q$ in $\mathrm{Proj}(M_2(A))$ and $p \sim_h q$ in $\mathrm{Proj}(M_4(A))$. Hence,

$$P(A)/\sim \simeq P(A)/\sim_h \simeq P(A)/\sim_u \ \text{ where } P(A) := \mathrm{colim}_n \, \mathrm{Proj}(M_n(A))$$

Relative K Theory and Excision

We repeat similar proofs in algebraic K-theory.

Corollary 2.7 *Suppose A is a C^* algebra, then the functors*

$$K_0, K_0 \circ M_n : C^* \,\mathrm{Alg} \to \mathrm{Ab}$$

$$M_n : C^* \,\mathrm{Alg} \to C^* \,\mathrm{Alg}, \quad A \mapsto M_n(A)$$

are naturally isomorphic for all n.

We extend to nonunital C^* algebras via

Definition 2.8 Suppose A is a C^* algebra, define

$$K_0(A) := \ker(K_0(A^+) \to \ker K_0(\mathbb{C}))$$

With respect to the topology, we have

Theorem 2.9 *(Homotopy invariance) Let A and B be C^* algebras and ϕ, ψ homotopic maps then*

$$\phi_* = \psi_* : K_0(A) \to K_0(B)$$

Proof: Let $F : A \to C([0,1], B)$ be given homotopy. Functorial composition of Proj, M_n and unitzation yields

$$F_n^+ : \mathrm{Proj}\, M_n(A^+) \to \mathrm{Proj}\, M_n(C[(0,1], B^+))$$

via identification $M_n(C([0,1], B^+)) \simeq C([0,1], M_n(B^+))$, a we obtain homotopy in matrices, and hence identity of ϕ_*^+, ψ_*^+. Nonunital case follows from restriction. □

Operator K-theory has yields nice description for relative K-theory. We explain the techniques involved and conclude with excision.

Lemma 2.10 *(Path lifting). The map $C([0,1], A) \to C([0,1], A/J)$ induces an isomorphism*

$$C([0,1], A)/C([0,1], J) \to C([0,1], A/J)$$

Proof: We have a $*$-injective morphism whose image contains dense $*$-subalgebra

$$\{f \cdot \pi(a) \;:\; f \in C([0,1]), a \in A\}$$

Now $*$-homomorphism have closed image. □

Proposition 2.11 *(Path lifting of unitaries and projections). Suppose u_t is a path of unitaries in A/J, $U \in U(A)$ ($\mathrm{Proj}(A)$) such that $\pi(U) = u_0$, then there is a lift of continuous paths of unitaries (projections) in $U(A)$($\mathrm{Proj}\, A$).*

Proof: We first prove the unitary case. By compactness, we may suppose

$$\sup_{t,s \in [0,1]} ||u_t^* u_s - 1|| < 1$$

Consider path $w_t := \ln u_0^* u_t$.This is well defined as $\ln$ is a holomorphic on union of their spectrum - a subset of $\sigma(\mathbb{T}) \setminus \{1\}$ using norm hypothesis. Denote lift W_t from Lem. 2.10. Define path (with inverse $e^{-W_t U^*}$),

$$Z_t := U e^{W^t}$$

and path via functional calculus

$$U_t := Z_t |Z_t|^{-1}$$

$U_0 = U$ and

$$\pi(U_t) = u_0 u_0^* u_t (u_t^* u_0^* u_0 u_t)^{-1} = u_t$$

□

Definition 2.12 A *relative K-cycle* for $(A, A/J)$ is a triple (p, q, x) where p and q are projections in $M_n(A)$ and x belongs to $M_n(A)$ and $\pi(x)$ is Murray-von Neumann equivalence between $\pi(p)$ and $\pi(q)$. If x itself is a Murray-von Neumann equivalence between p and q we say that the relative K-cycle is *degenerate*.

For example. Let $T \in \mathcal{L}(H)$, so by Atkinson's lemma, $T^*T - I, TT^* - I$ are compact. With $A = \mathcal{L}(H)$, $J = \mathcal{K}(H)$, and A/J the Calkin algebra. Then (I, I, T) is a K-cycle.

We now have a group.

Definition 2.13 The *relative K-group* $K_0(A, A/J)$ is defined to be the group with one generator $[p, q, x]$ for each relative K-cycle (p, q, x) with relations:

1. If (p_0, q_0, x_0) and (p_1, q_1, x_1) are relative k-cycles that can be connected by a continuous paths of relative cycles.
2. if (p, q, x) is degenerate then $[p, q, x] = 0$.
3. $[p \oplus p', q \oplus q', x \oplus x'] = [p, q, x] + [p', q', x']$ for all relative cycles (p, q, x) and (p', q', x').

Proofs in relative K-theory are easy as there are useful paths: let (p, q, x) be a relative cycle, u_t a path of unitaries, then

$$(u_t^* p u_t, u_t^* q u_t, u_t^* x u_t), \quad (p, u_t^* q u_t, u_t^* x)$$

is a path of relative cycles; if $\pi(p) = \pi(x)$, then $\pi(q) = \pi(x)$ and

$$(p, q, tp + (1 - t)x)$$

is a path of cycles. Applying this, we see $K_0(A, A/J)$ is an abelian group with

$$[(q, p, x^*)] = -[(p, q, x)]$$

Hence, from Prop. 4.6.

Corollary 2.14 *(Half exactness). The sequence*

$$K_0(A, A/J) \to K_0(A) \to K_0(A/J)$$

induced by the forgetful map $f : [(p, q, x)] \mapsto [p] - [q]$ is exact.

By mentioned techniques, the natural map

$$K_0(J) \simeq K_0(J^+, \mathbb{C}) \to K_0(A, A/J)$$

is an isomoprhism, giving

Corollary 2.15 *The SES of C^*-algebras*

$$0 \to J \to A \to A/J \to 0$$

yields half exact sequence

$$K_0(J) \to K_0(A) \to K_0(A/J)$$

Stability and Morita Invariance

We give an example of unviersal C^* algebras [5] and prove stability axiom.

Theorem 2.16 *Suppose that $\{(A_n, \varphi_n)\}$ is a direct sequence of C^* algebras. Then the direct limit $(A, \varphi^n) = \varinjlim(A_n, \varphi_n)$ always exists.*

Proof: An explicit construction is a subspace of $\prod_{i\geq 0} A_i$ given by

$$\mathcal{A} := \left\{ (a_i)_{i\in\mathbb{Z}_{\geq 0}} \in \prod_i A_I \,:\, \exists i_0 \geq 0, a_i = \phi_{i,j}(a_j) \forall i > j > i_0 \right\}, \quad ||(a_i)|| = \limsup_i ||a_i||_{A_i}$$

where $\phi_{i,j} := \phi_{i-1} \circ \cdots \circ \phi_j$. [6] Completion of seminorm yields universal object. □

Corollary 2.17 *Let H be an infinite dimensional separable Hilbert space, and $p_n \in \mathcal{L}(H)$ a sequence of finite increasing rank projections. Then we have*

$$\varinjlim \mathcal{L}(p_n H) \cong \mathcal{K}(H)$$

in particular,

$$\varinjlim M_n(\mathbb{C}) \simeq \mathcal{K}$$
$$A \otimes \mathcal{K} \simeq \varinjlim M_n(A)$$

We have a special result for direct limits, we prove surjectivity and refer the rest in [HR00, Prop. 4.1.15] (which uses similar tricks).

Theorem 2.18 *Let A be a unital C^* algebra and suppose there is an increasing sequence*

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A$$

of unital C^* algebras whose union is dense A. Then the induced map

$$\varinjlim K_0(A_j) \to K_0(A)$$

is an isomoprhism. Hence, $K_0(\mathcal{K}) \simeq K_0(\mathbb{C})$.

[5] There are simple conditions [NdK17, Prop 5.15], to resolve the nonexistence of "free objects" in C^* Alg.

[6] ϕ_i are $*$-homomorphisms, hence norm decreasing so seminorm makes sense.

Proof: Let $p \in M_n(A)$ be projection. Exists some j, self adjoint $a \in M_n(A_j)$, $||a - p|| < 1$, by approximate identity. Hence, applying functional calculus on $\sigma(a) \subseteq (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$, with

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda < 1/2 \\ 1 & \text{if } \lambda > 1/2 \end{cases}$$

$f(a)$ is a projection in $M_n(A_j)$, $||f(a) - a|| < \frac{1}{2}$. So $||q - p|| < 1$. Thus, p, q are unitarily equivalent, [HR00, Prop. 4.1.7], and p lies in image. □

Stability axioms leads to the notion of Morita equivalence in C^* algebras.

Theorem 2.19 *[Bro73]. Let A and B be separable C^* algebras. Then they are strongly Morita equivalent if and only if $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$.* [7]

Morita equivalence yields same K-, E- and KK-theory. An important application is

Corollary 2.20 *If $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert A modules and if*[8] $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle = A$, *then the natural inclusion*

$$\mathcal{K}_A(\mathcal{E}_1) \to \mathcal{K}_A(\mathcal{E}_1 \oplus \mathcal{E}_2)$$

induces isomoprhism [9]

$$K(\mathcal{K}_A(\mathcal{E}_1)) \to K(\mathcal{K}_A(\mathcal{E}_1 \oplus \mathcal{E}_1))$$

This follows as the C^* algebras $\mathcal{K}_A(\mathcal{E}_1)$ and $\mathcal{K}_A(\mathcal{E}_1 \oplus \mathcal{E}_2)$ are Morita equivalent. A special case occurs when H_1 and H_2 are Hilbert spaces, the inclusion[10] $\mathcal{K}(H_1) \subseteq \mathcal{K}(H_1 \oplus H_2)$ induces isomoprhism

$$\mathcal{K}(A \otimes \mathcal{K}(H_1)) \hookrightarrow K(A \otimes \mathcal{K}(H_1 \oplus H_2))$$

for every C^* algebra A.

2.2 C^* Algebra Homology Theory

We follow [Fra18] and look into the axioms of operator K-theory.

Definition 2.21 A *homology theory* for C^* algebras is a sequence of functors $\{E_n : C^* \mathrm{Alg} \to \mathrm{Ab}\}$, $n \leq 0$ which are $*$-homotopic invariant and for each SES of C^* algebras $0 \to I \to A \to B \to 0$ there is a natural LES

$$\begin{array}{ccccc} E_0(I) & \longrightarrow & E_0(A) & \longrightarrow & E_0(B) \\ & & \partial & & \\ E_{-1}(1) & \longrightarrow & E_{-1}(A) & \longrightarrow & E_{-1}(B) \\ & & \partial & & \\ E_{-2}(I) & \longrightarrow & E_{-2}(A) & \longrightarrow & E_{-2}(B) \end{array}$$

Lemma 2.22 *Let E be a half exact functor from C^* algebras to abelian group. Then,*

[7] This is also called stably equivalent.
[8] This condition means that $\mathcal{E}$ is *full*. A condition enjoying many properties.
[9] $\mathcal{K}_A(\mathcal{E})$ for a Hilbert A-module is a closed ideal of its adjointable morphisms, in particular a C^* algebra.
[10] $\mathcal{K}(H)$ here denotes the compact operators on Hilbert space.

1. *For any C^* algebras A_1, A_2, the $E(A_1 \oplus A_2) \cong E(A_1) \oplus E(A_2)$ under the induced standard injections and projectinos.*

2. *If α and β are such that $\alpha + \beta$ is also a $*$-homomoprhism, then $E(\alpha+\beta) = E(\alpha) + E(\beta)$.*

Proof: For 1, half exactness and functoriality implies in the category Ab, we have

$$E(A_1) \xrightarrow{E(i_1)} E(A_1 \oplus A_2) \xrightarrow{E(p_2)} E(A_2), \qquad E(A_2) \xrightarrow{E(i_2)} E(A_1 \oplus A_2)$$

where $E(i_2)E(p_2) = 1$ similarly for the other pair $E(i_2), E(p_1)$. This characterizes $E(A_1 \oplus A_2)$. 2 follows as $\alpha(A), \beta(A)$ are $*$-subalgebras. We apply 1 to the copy $\alpha(A) \oplus \beta(A)$. $\square$

The Barrat Puppe Sequence

Through standard topological constructions of cones and suspensions, we show how to recover the connecting homomorphism. Conversely, we construct a homology theory via Barrat Puppe sequence for any half exact, homotopy invariant functor.

Let A be a C^* algebra. The *cone* over A, written $CA := C_0((0,1], A]$ is the C^* algebra of continuous functions $f : [0,1] \to A$ vanishing at 0. The *suspension* of A, $SA := C_0((0,1), A)$. S and C are covariant functors on C^* Alg. [11] CA is contractible with homotopy

$$H : CA \to C([0,1], CA), \quad t \mapsto f_{1-t}(s) = \begin{cases} 0 & \text{if } s \leq t \\ f(s-t) & \text{if } t < s \leq 1 \end{cases}$$

Corollary 2.23 *Naturality of LES from*

$$0 \to SA \to CA \to A \to 0$$

yields natural isomoprhisms $\epsilon : E_{n-1} \xrightarrow{\simeq} E_n \circ S$.

For a SES

$$0 \to I \to A \xrightarrow{\pi} B \to 0$$

The *mapping cone C_π* of the is the C^* algebra

$$C_\pi := \{(f, a) \in CB \oplus A \ : \ f(1) = \pi(a)\}$$

$$0 \to I \to A \xrightarrow{\pi} B \to 0$$

Proposition 2.24 *The boundary maps $\partial : E_{n-1}(B) \to E_n(I)$, $n \leq 0$ are given by*

$$\partial = -E_n(i_2)^{-1} \circ E_n(i_1) \circ \epsilon_B$$

where i_1, i_2 are inclusions of SB, I into C_π and $\epsilon_B : E_{n-1}(B) \xrightarrow{\simeq} E_n(SB)$ is the boundary map of $0 \to SB \to CB \to B \to 0$.

[11] From nuclearity, one deduce $CA \cong C_0((0,1]) \otimes A$ and $C_0((0,1)) \otimes A \cong C_0(\mathbb{R}, A)$.

Proof: We have commutative diagram with exact rows, where $\iota = [i_1,\ i_2]$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & SB & \longrightarrow & CB & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle pr_1} & & \uparrow{\scriptstyle pr_1} & & \| & & \\
0 & \longrightarrow & SB \oplus I & \xrightarrow{\iota} & C_\pi & \xrightarrow{p} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle pr_2} & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\pi} & B & \longrightarrow & 0
\end{array}$$

From naturality of LES,

$$\begin{array}{ccc}
E_{n-1}(B) & \xrightarrow{\epsilon_B} & E_n(SB) \\
\| & & \uparrow{\scriptstyle E_n(pr_1)} \\
E_{n-1}(B) & \xrightarrow{d} & E_n(SB \oplus I) \\
\| & & \downarrow{\scriptstyle E_n(pr_2)} \\
E_{n-1}(B) & \xrightarrow{\partial} & E_n(I)
\end{array}$$

Combining with Cor. 2.22

$$\begin{array}{ccccc}
 & & E_n(CB) & & \\
 & & \uparrow & & \\
E_{n-1}(B) & \xrightarrow{d=\begin{bmatrix}\epsilon_B \\ \partial\end{bmatrix}} & E_n(SB \oplus I) \simeq E_n(B) \oplus E_n(I) & \xrightarrow{E_\iota = [E_n(i_1),\ E_n(i_2)]} & E_n(C_\pi) \\
 & & \downarrow & & \\
 & & E_n(I) & &
\end{array}$$

$E_n(\iota)d = 0$ then inmplies $E_n(i_1)\epsilon_B + E_n(i_1)\partial = 0$. □

Prop. 2.24 suggests the converse construction. We need a lemma first.

Lemma 2.25 *Let $E : C^* \operatorname{Alg} \to \operatorname{Ab}$ be a homotopy invariant half exact functor, then given SES, $0 \to I \to A \xrightarrow{\pi} B \to 0$, $i_2 : I \to C_\pi$ induces an isomorphism $E(I) \to E(C_\pi)$.*

Proof: Surjectivity follows from SES $0 \to I \to C_\pi \to CB \to 0$. Construct auxiliary objects.

$$\begin{aligned}
&Q := \{f \in C([0,1], A) \ : \ f(0) \in I\} \\
&\alpha : I \to Q, \quad b \mapsto (c_b : t \mapsto b) \\
&\beta : Q \to I, \quad f \mapsto f(0) \\
&\gamma : Q \to C_\pi, \quad f \mapsto (\pi f, f(1)) \\
&\alpha\beta \simeq id_Q, \beta\alpha = id_I
\end{aligned}$$

$E(\gamma)$ is injective from $0 \to CI \to Q \xrightarrow{\gamma} C_\pi \to 0$. So $E(i_2)$ is injective as

$$\begin{array}{ccc}
I & \xrightarrow{i_2} & C_\pi \\
\downarrow{\scriptstyle \alpha} & \nearrow{\scriptstyle \gamma} & \\
Q & &
\end{array}$$

□

Theorem 2.26 *Let $E : C^* \operatorname{Alg} \to \operatorname{Ab}$ be homotopy-invariant, half exact functor. Define $E_{-n}, n \geq 0$ by $E_{-n} := E \circ S^n$ and for SES of C^* algebras $0 \to I \to A \xrightarrow{\pi} B \to 0$, define*

$$\partial : E_{n-1}(B) = E_n(SB) \to E_n(I), \quad \partial = -E_n(i_2)^{-1} \circ E_n(i_1)$$

where $i_1 : SB \to C_\pi, i_2 : I \to C_\pi$ are component inclusions. This forms a C^* algebra homology theory.

Proof: E_ns are homotopy invariant half exact as suspension preserves exactness and $*$-homotopies. Boundary maps are natural as construction of mapping cone is. We show LES is exact:

$$E_n(SA) \to E_n(SB) \xrightarrow{\partial} E_n(I) \to E_n(A) \to E_n(B)$$

E_n is half exact so we have exactness at $E_n(A)$. From

$$\begin{array}{ccccc} E_n(SB) & \xrightarrow{\partial} & E_n(I) & \longrightarrow & E_n(A) \\ \downarrow{\scriptstyle -id} & & \downarrow{\scriptstyle E_n(i_2)} & & \| \\ E_n(SB) & \xrightarrow{E_n(i_1)} & E_n(C_\pi) & \xrightarrow{E_n(pr_2)} & E_n(A) \end{array}$$

From Lem. 2.25 we get exactness at $E_n(I)$.

Lastly, we construct auxiliary double cone with an associated SES

$$D = \{(f, g) \in CB \oplus CA \, : \, f(1) = \pi(g(1))\}$$

$$0 \to SA \xrightarrow{i_2} D \xrightarrow{q} C_\pi \to 0, \quad q : (f, g) \mapsto (f, f(1))$$

This yields commutative diagram,

$$\begin{array}{ccccc} E_n(SA) & \xrightarrow{E_n(S\pi)} & E_n(SB) & \xrightarrow{\partial} & E_n(I) \\ \| & & \downarrow{\scriptstyle -E_n(i_1)} & & \downarrow{\scriptstyle E_n(i_2)} \\ E_n(SA) & \xrightarrow{E_n(i_2)} & E_n(D) & \xrightarrow{E_n(q)} & E_n(C_\pi) \end{array}$$

From Lem. 2.22, reversal map $r : SA \to SA$ induces $-id$. Also, $i_2 \circ r, i_1 \circ S\pi : SA \to D$ are $*$-homotopic. This gives exactness at $E_n(SB)$, by Lem. 2.25. □

2.3 Cuntz' Proof of Bott Periodicity

Operator K Theory is in fact a *stable* C^* homology theory. We regard $\mathcal{K}$ as algebra of compact operators $l^2(\mathbb{N})$, and denote E_{ij} the rank 1-operators sending ith to jth standard basis.

Definition 2.27 A functor $E : C^* \operatorname{Alg} \to \operatorname{Ab}$ is *stable* if the corner inclusion [12] inclusion $E_{00} \otimes id_A : A \to \mathcal{K} \otimes A$, induces an isomoprhism $E(A) \xrightarrow{\cong} E(\mathcal{K} \otimes A)$.

[12]The choice of E_{00} is arbitrary, it can be replaced by any other E_{ij}

We prove that all stable C^* algebra homology theory satisfies Bott periodicity. The argument has a nice geometric picture, using the notion of *infinite swindle*.

Definition 2.28 (Toeplitz algebra) The isometry S,

$$S : \mathcal{K} \to \mathcal{K}, \quad \xi_n \mapsto \xi_{n+1}$$

generates a C^* algebra, $C^*(S) := \mathcal{T}$, the *Toeplitz algebra*.

The Toeplitz algebra $C^*(S)$ is the universal unital C^* algebra generated by an isometry, [Cob67]: for any unital $A \in C^*\,\mathrm{Alg}$ and any isometry $w \in A$, there is a unique unital[13] $*$-homomorphism,

$$C^*(S) \xrightarrow{S \mapsto w} A$$

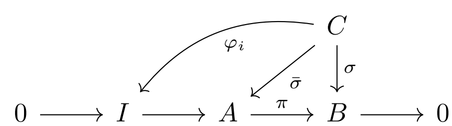
Definition 2.29 We denote $\sigma : \mathcal{T} \to C(S^1)$, $S \mapsto z$, the *symbol map*. The kernel of σ is the compact operators $\mathcal{K}$, yields the *Toeplitz extension*[14]

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\sigma} C(S^1) \to 0$$

we let σ_1 be the composition of σ with the evaluation map at 1.

We will require the small lemma, [Fra18, Lem. 12]

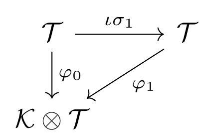
Lemma 2.30 *Suppose $0 \to I \to A \xrightarrow{\pi} B \to 0$ is a SES. Given $\varphi_i : C \to I$, $i = 0, 1$ and $\sigma : C \to B$ $*$-homomoprhisms and σ has a lift $\bar{\sigma} : C \to A$.*



If $\theta_i = \varphi_i + \bar{\sigma}$, $i = 0, 1$ are $*$-homomoprhisms [15]*, and exists homotopy between θ_0 and θ_1 such that $\pi \circ \theta_t = \sigma, \quad \forall t \in [0, 1]$, then φ_0, φ_1 induces the same maps in homology.*

Theorem 2.31 *Let E_{-n}, $n \geq 0$ be a stable C^* algebra homology theory. Then, the maps $\sigma_1 : \mathcal{T} \to \mathbb{C}$, $\iota : \mathbb{C} \to \mathcal{T}$ induces inverse isomorphisms between $E_{-n}(\mathcal{T})$ and $E_{-n}(\mathbb{C})$.*

Proof: [16] We do an *infinite swindle* by working in larger spaces. Regard $\mathcal{K} \otimes \mathcal{T}$ as operators on $l^2(\mathbb{N}) \otimes l^2(\mathbb{N})$ and define



$$\varphi_0 : S \mapsto E_{00} \otimes S, \quad \varphi_1 :, \quad S \mapsto E_{00} \otimes 1$$

By stability, result follows if $E_n(\varphi_0) = E_n(\varphi_1)$. We work in an auxiliary space. Consider SES,

$$0 \to \mathcal{K} \otimes \mathcal{T} \to \mathcal{K} \otimes \mathcal{T} + \mathcal{T} \otimes 1 \xrightarrow{\pi} C(S^1) \to 0$$

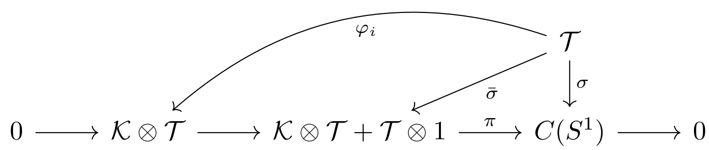
[13] Note that $\mathcal{T}$ is unital

[14] The proof is an application of functional calculus with Atkinson's lemma, [HR00, Ch.2]

[15] we identified φ_i in A

[16] One might show $\psi_1 := \iota \circ \sigma_1$ is homotopic to identity, but this is not true.

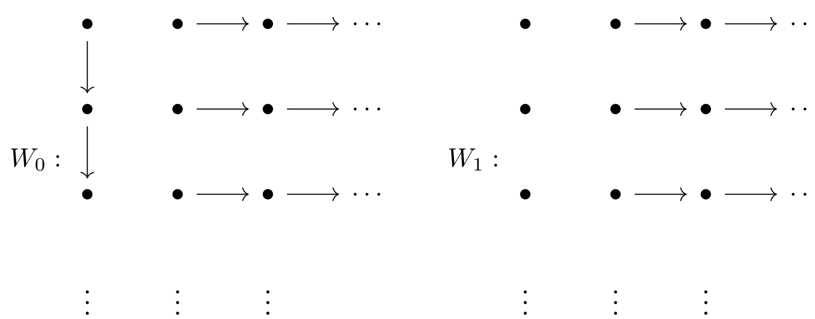
where first map inclusion, second map is restriction to $\mathcal{T}\otimes 1, \quad S\otimes 1 \mapsto z$. We apply Lem. [Fra18, Lem. 12], to the diagram



$$\bar{\sigma} : S \mapsto S(1-E_{00})\otimes 1$$

To apply lemma, define isometries W_0, W_1 in $\mathcal{K}\otimes\mathcal{T}+\mathcal{K}\otimes 1$ by

$$W_0 = E_{00}\otimes S + S(1-E_{00})\otimes 1, \quad W_1 = E_{00}\otimes 1 + S(1-E_{00})\otimes 1$$



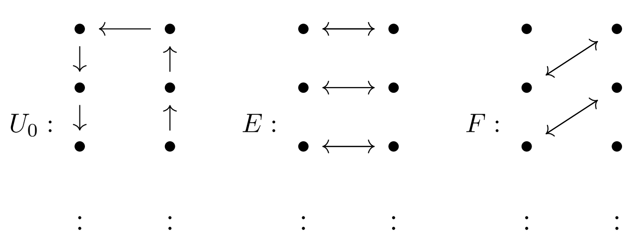
$\downarrow, \rightarrow$ are increasing direction

So action of S goes downwards. We construct a path of isometries W_t, $t\in[0,1]$ from W_0 to W_1 satisfying $\pi(W_t) = z = \pi(S\otimes 1)$ for $t\in[0,1]$.

The construction begins in $M_2(\mathbb{C})\otimes\mathcal{T}$. Let

$$U_0 = E_{00}\otimes S + E_{10}\otimes E_{00} + E_{11}\otimes S^*$$

$$E = E_{01}\otimes 1 + E_{10}\otimes 1, \quad F = E_{00}\otimes E_{00} + E_{01}\otimes S^* + E_{10}\otimes S$$



$\downarrow, \rightarrow$ are increasing direction

So $U_0 = FE$ is homotopic to identity in unitaries of $M_2(\mathbb{C}) \otimes \mathcal{T}$.[17]

We define a new homotopy V_t,by adding $(1-E_{00}-E_{11})\otimes 1 \in \mathcal{T}\otimes 1$ to the path in $M_2(\mathbb{C})\otimes\mathcal{T}$, from $V_0 = U_0 + (1 - E_{00} - E_{11}) \otimes 1$ to $V_1 = 1$. Since $\pi(1 - E_{00} - E_{11}) = \pi\Big((SS^*)^2\Big)$, $\pi(V_t)\pi(W_1) = z$ for all t. We have our homotopy. □

With the same proof as Thm. 2.31, but tensoring a C^* algebra A, and id_A at the right place,

Corollary 2.32 *For any C^* algebra A, we get isomorphisms between $E_{-n}(\mathcal{T} \otimes A)$ and $E_{-n}(A)$.*

Now let $\mathcal{T}_0$ be the kernel of the symbol map $\sigma_1 : \mathcal{T} \to \mathbb{C}$, then there is a split short exact,

$$0 \longrightarrow \mathcal{T}_0 \longrightarrow \mathcal{T} \underset{\curvearrowleft \iota}{\xrightarrow{\sigma_1}} \mathbb{C} \longrightarrow 0$$

Applying previously results, for a stable homology theory, one has $E_n(\mathcal{T}_0 \otimes A) = 0$, $n \leq 0$ for any C^* algebra A, in particular

Theorem 2.33 *(Bott Periodicity) For any C^* algebra A, the boundary maps in extension $0 \to \mathcal{K} \otimes A \to \mathcal{T}_0 \otimes A \xrightarrow{\sigma_0 \otimes id} SA \to 0$ are natural isomoprhisms, $E_{n-2}(A) \to E_n(A)$, having made the identifications, $E_{n-1}(SA) = E_{n-2}(A)$, $E_n(\mathcal{K} \otimes A) = E_n(A)$.*

Proof: We have the SES of nonuntial form of Toeplitz extension,

$$0 \to \mathcal{K} \to \mathcal{T}_0 \xrightarrow{\sigma_0} C_0(0,1) \to 0$$

σ_0 is symbol map with $(0,1)$ identified to $S^1 \setminus \{1\}$ via $t \mapsto e^{2\pi i t}$. As $C_0(0,1)$ is nuclear, tensoring with A remains exact, [PBO08, Cor. 3.7.4],

$$0 \to \mathcal{K} \otimes A \to \mathcal{T}_0 \otimes A \xrightarrow{\sigma_0 \otimes id} SA \to 0$$

Result follows from Cor. 2.32. □

Corollary 2.34 *Given a SES, $0 \to I \to A \to B \to 0$, there is a six term exact sequence*

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ \uparrow & & & & \downarrow {\scriptstyle \partial_2 \circ \partial} \\ K_{-1}(B) & \longleftarrow & K_{-1}(A) & \longleftarrow & K_{-1}(I) \end{array}$$

where ∂ is homomoprhism $K_0(B) \to K_{-2}(B)$ induced from Thm. 2.33, and ∂_2 from LES.

[17] If G is idempotent unitary in C^* algebra, $G = -P + P^\perp, P^\perp = 1 - P$, for P projection. This yields a homotopy to identity via unitary path $t \mapsto e^{\pi i t} P + P^\perp$.

Chapter 3
Orientations and Chern Character

In this chapter we set up the topological background for topological index and the Chern character map. The first section describes orientations and fiber integration - the general construction which can be omitted in first read. It is strongly recommended to be reviewed after reading Chapter 4. The last three sections build from scratch the Chern character map, using tools as the Serre spectral sequence.

R will denote unital commutative ring. We introduce Serre Spectral Sequence to compute $H^*(BU(n); R)$, bringing us to characteristic classes. All the spaces X are paracompact. We use $H^*(X; R) := \prod H^q(X; R)$.

3.1 Universal E-Orientations

Let $V \to X$ be a vector bundle, we associate to it a metric[1]. Define $\mathrm{Th}(V) := D(V)/S(V)$, the *Thom space* of V, as the quotient of the total spaces of disk bundle by the sphere bundle, .

Definition 3.1 Let E be a multiplicative cohomology theory, $V \to X$ a topological vector bundle of rank n. An *E-orientation* or *E-Thom class* on V is an element of degree n,

$$u \in \tilde{E}^n(\mathrm{Th}(V)) \cong E(D(V), S(V))$$

in the reduced E-cohomology[2] such that each restriction $j_x^* u \in E^n(D(V)_x, S(V)_x) \cong \tilde{E}^n(S^n)$ is a generator, where j_x denotes the inclusion map of each fiber.

Right conditions yields *Thom-Dold isomoprhism*, $c \cup (-) : E^*(X) \xrightarrow{\simeq} \tilde{E}^{*+n}(\mathrm{Th}(V))$, [Koc96, Prop 4.3.6]. In Ch. 4, we compare Thom homomorphisms of KU and ordinary cohomology theory. This is the motivation for our topological index map in Thm. 4.12. It is computed through *fiber integration* via orientation.

Definition 3.2 (Pontrjagin Thom collapse map) Let $i : X \hookrightarrow Y$ be a tubular embedding embedding of manifolds. Denote the normal bundle as the fiberwise quotient $N_i X := i^*TY/TX$.

$$c_i : Y \to Y/(Y - f(N_i X) \xrightarrow{\simeq} \mathrm{Th}(N_i X)$$

[1] such exists as X is paracompact

[2] one may construct a reduced from an unreduced theory, and vice versa [ASGP02, Ch 12.].

Observe that if B is a compact Hausdorff space, then

$$(B \times \mathbb{R}^n)^+ \cong \Sigma^n B_+ \cong S^n \wedge B_+$$

where $(-)^+$ denotes one point compactification and $(-)_+$ the adjoining a base point.

This yields an algorithm. Let H be a multiplicative cohomology theory, $p : E \to B$ a fiber bundle over compact space, and a tubular embedding,

$$i : E \hookrightarrow B \times \mathbb{R}^n$$

$$N_i E \to E$$

The Thom class map factors to a map τ.

$$\begin{array}{ccc} B \times \mathbb{R}^n & \longrightarrow & \mathrm{Th}(N_i(E)) \\ & \searrow & \nearrow_{\tau} \\ & (B \times \mathbb{R}^n)^+ & \end{array}$$

by universal property. Suppose $\mathrm{Th}(N_i E)$ has an H-orientation and a Thom isomoprhism,

$$\begin{array}{ccc} H^{*+n-\dim F}(\mathrm{Th}(N_i E)) & \xrightarrow{\tau} & \tilde{H}^{*+n-\dim F}(\Sigma^n B_+) \\ \uparrow{\scriptstyle thom} & & \downarrow{\scriptstyle susp, \simeq} \\ H^*(E) & & H^{*-\dim F}(B) \end{array}$$

This is the map we will see in Ch. 4.

3.2 Serre Spectral Sequence

We elaborate on parts of [Koc96].

Theorem 3.3 *(Serre Spectral Sequence) Let*

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

be a Serre fibration. Assume B is a simply connected simplicial complex[3]*. There is a multiplicative spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(F; R)) \Rightarrow H^*(E; R)$$

[3] There is another version for R charecteristic 2. Proof is verbatim.

Proof: Let $\{B^n\}_{n\in\mathbb{N}}$ be n-skeletal filtration of B; $E^n = \pi^{-1}(B^n)$. LES in cohomology induces exact couple

$$\begin{array}{ccc} H^*(E^n;R) & \xrightarrow{\quad j^* \quad} & H^*(E^{(n-1)};R) \\ {\scriptstyle p^*}\nwarrow & & \swarrow{\scriptstyle \delta} \\ & H^*(E^n, E^{(n-1)};R) & \end{array}$$

The bigrading is

$$D^{s,t} = H^{s+t}(E^s;R), \quad E_1^{s,t} = H^{s+t}(E^{s+t}, E^{(s+t-1)};R)$$

The convergence condition [Koc96, Lem. 2.2.4] is: *for fixed k,*

$$H^k(E^n) \to H^k(E^{n-1})$$

is an isomorphism for all n sufficiently large.

Using cellular cohomology, the statement is true for filtration $\{B^n\}_{n\in\mathbb{N}}$,

$$H^k(B^n) \to H^k(B^{n-1})$$

for all n large. As we have serre fibration, we show this property passes to total space.

By cellular approximation, [Hat02, Thm. 4.8], $\pi_i(B, B^n) = 0$ for sufficiently large n. If (B, B^n) is $k-1$-connected, then so is (E, E^n) by lifting property. By Hurewicz theorem and LES $H_k(E^n;\mathbb{Z}) \to H_k(E;\mathbb{Z})$ is an isomoprhism, hence by naturality of UCT,

$$0 \to \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), R) \to H^n(X;R) \to \mathrm{Hom}(H_n(X;\mathbb{Z});R) \to 0$$

$H^i(E;R) \to H^i(E^n;R)$ is an isomoprhism. In particular, the filtration E^n satisfies Mittag-Leffler condition, [Koc96, Cor. 4.2.4], so spectral sequence converges to[4]

$$H^*(E) = \varprojlim H^*(E^n).$$

We describe the E_1 page. Let $I_n := \{\ \Delta : n\text{- simplix of } B\ \}$. Subdividing if necessary, assume $(\pi^{-1}(i), \pi^{-1}(\partial(i))) \simeq (D^n, S^{n-1}) \times F$, for each $i \in I_n$. Thus,

$$\begin{aligned} E_1^{n,t} &= H^{n+t}(E^n; E^{n-1}; R) \simeq \tilde{H}^{n+t}(E^n/E^{n-1}; R) \\ &= \tilde{H}^{n+t}\left(\bigvee S^n \wedge F_+; R;\right) \simeq \prod_{i\in I_n} \tilde{H}^t((F_i)_+; R) \\ &\simeq \prod_{i\in I_n} H^t(F_i;R) = C^n(B; H^t(F_i;R)) \end{aligned}$$

by wedge[5] and suspension axiom. Let B_i^n denote interior of $i \in I_n$ removed. $f_j : S^n \to B^n$ attaching map. Define, for skeletal filtration $\{B^n\}_{n\in\mathbb{N}}$,

$$f_{ji} : S^n \xrightarrow{f_j} B^n \to B^n/B_i^n \simeq B_i^n, \quad d_{ji} := \deg f_{ji}$$

[4]Here we have direct product of cohomology.

[5]If for any set of pointed CW complexes X_i, $\tilde{H}^*(\bigvee_i X_i) \simeq \prod_i \tilde{H}^*(X_i)$ via canonical map.

Cellular complex of B^n is defined[Hat02, Ch. 2] such that

$$\partial : \bigoplus_{j \in I_{n+1}} \mathbb{Z}e_j \xrightarrow{\partial_n} \bigoplus_{i \in I_n} \mathbb{Z}e_i, \quad \partial\left(\sum_j n_j e_j\right) = \sum_i \left(\sum_j d_{ji} n_j\right) e_i$$

Computing d_1,

$$d_1 : E_1^{s,t} = H^{s+t}(E^s, E^{s-1}; R) \to H^{s+t}(E^s; R) \xrightarrow{\delta} H^{s+t+1}(E^{s+1}, E^s; R) = E_1^{s+1,t}$$

$$\begin{array}{ccc}
\mathrm{Hom}\left(\bigoplus_{i \in I_n} \mathbb{Z}e_i; H^t(F;R)\right) & \xrightarrow{\mathrm{Hom}(\partial,1)} & \mathrm{Hom}\left(\bigoplus_{j \in I_{n+1}} \mathbb{Z}e_j; H^t(F;R)\right) \\
\downarrow \simeq & & \downarrow \simeq \\
E_1^{s,t} \simeq \prod_{i \in I_n} H^t(F;R) & \xrightarrow{d_1} & \prod_{j \in I_{n+1}} H^t(F_j;R) \simeq E_1^{s+1,t} \\
\downarrow & & \downarrow \\
H^t(F_i;R) & \xrightarrow{d_{ij}} & H^t(F_j;R)
\end{array}$$

shows d_1 coincides with cellular cohomology of B with coefficients in $H^t(F;R)$. This gives the E_2 page. The multiplicative property follows as AHSS [Koc96, Prop. 4.2.9], and coincides with the cup product. [6] □

Proposition 3.4 *(Thom Gysin sequence) Given a Serre Fibration*

$$\begin{array}{ccc}
S^n & \longrightarrow & E \\
 & & \downarrow \pi \\
 & & B
\end{array}$$

Assume that B is a simply connected simplicial complex. Then there is a long exact sequence

$$\cdots \to H^k(B;R) \xrightarrow{\pi^*} H^k(E;R) \to H^{k-n}(B;R) \xrightarrow{\tau} H^{k+1}(B;R) \to \cdots$$

where τ is given by an element $u \in H^{n+1}(B;R)$, called the Euler class of π.

Proof: The conditions of Thm. 3.3, are satisfied. As $H^t(S^n;R) = R$ only when $t = 0, n$, nontrivial differentials occur at,

$$E_{n+1}^{*,n} \xrightarrow{d_{n+1}} E_{n+1}^{*+n+1,0}$$

[6] A simpler proof for multiplicative property is via Cartan Eilenberg system, [Goe].

Let $i := 1 \in H^0(B,R) \cong E_{n+1}^{0,n}$. The multiplicative structure, denoted $\cdot$, induces

$$E_{n+1}^{*,0} \to E_{n+1}^{*,n}, \quad i \otimes b \mapsto i \cdot b$$

This is an isomorphism, as multiplicative structure is given by cup product, [Koc96, Ch. 4]. Let $u := d_{n+1}(i) \in E_{n+1}^{n+1,0}$. As d_{n+1} is a graded derivation, d_{n+1} is given by cup product with u.

$$d_{n+1}(i \cdot b) = d_{n+1}(i) \cdot b + (-1)^n i \cdot d_{n+1}(b) = u \cdot b$$

From the following diagonal

$$\begin{array}{ccc} \ddots & & \\ \nwarrow & & \\ & E_\infty^{s-1,1} & \\ & \nwarrow & \\ & & E^{s,0} \end{array}$$

We have

$$E_\infty^{s,0} \xrightarrow{\simeq} F^s H^s \xrightarrow{\simeq} \cdots \to F^{s-n} H^s \to E_\infty^{s-n,n} \to 0.$$

As $E_\infty^{s-n-k,n+k} = \{0\}$ for all $k \geq 0$, implies, $F^{s-n}H^s = H^s(E;R)$, which yields,

$$0 \to E_\infty^{s,0} \to H^s(E;R) \to E_\infty^{s-n,n} \to 0$$

Combining with exact sequence,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_\infty^{s,n} & \xrightarrow{\ker d_{n+1}} & E_{n+1}^{s,n} & \xrightarrow{d_{n+1}} & E_{n+1}^{s+n+1,0} & \xrightarrow{\operatorname{coker} d_{n+1}} & E_\infty^{s+n+1,0} \longrightarrow 0 \\ & & & & \downarrow \simeq & & \downarrow \simeq & & \\ & & & & H^s(B;R) & \xrightarrow{u \cdot (-)} & H^{s+n+1}(B;R) & & \end{array}$$

yields our desired equation. $\square$

3.3 Cohomology of $BU(n)$

We look at $G = U(n)$ and at a model of the classifying space of $BU(n)$.[7] The section not only computes cohomolgy, but set up the bijection between characteristic classes that would be used to define Todd classes.

Definition 3.5 $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $G_n(\mathbb{F}^q)$ be the *Grassmanian manifolds*. and $E_n(\mathbb{F}^q)$ the canonical n vector bundle over $G_n(\mathbb{F}^q)$.

$$G_n(\mathbb{F}^\infty) := \operatorname{colim} G_n(\mathbb{F}^q), E_n(\mathbb{F}^\infty) := \operatorname{colim} E_n(\mathbb{F}^q).$$

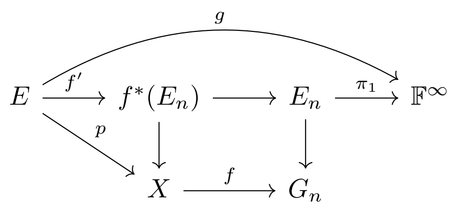
be direct limit of spaces via canonical inclusion. These form the *universal bundle* $\gamma_n^{\mathbb{F}} = (E_n(\mathbb{F}^\infty), G_n(\mathbb{F}^\infty), \pi)$.

Theorem 3.6 *For paracompact spaces X, we have a natural bijection*

$$[X, G_n(\mathbb{F}^\infty)] \to \mathrm{Vect}_{\mathbb{F}}^n(X), \quad [f] \mapsto f^*(E_n(\mathbb{F}^\infty))$$

[7] Similar argument follows with for $G = O(n)$ for $\mathbb{Z}/2$ coefficients

Proof: Isomorphisms $E \cong f^*(E_n)$ of $E \to X$ corresponds to liner injections $g : E \to \mathbb{F}^\infty$.



where $\pi_1(l, v) = v$. Given isomorphism, composition is injective. Conversely, given g, an isomorphism is given by $g(p^{-1}(x))$. Surjectivity follows by piecing local trivializations, injectivity by a "switch" homotopy: let $g_0, g_1 : E \to \mathbb{F}^\infty$. Define $L_t^+ : \mathbb{F}^\infty \to \mathbb{F}^\infty$

$$(x_1, x_2, \ldots) = (1 - t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, 0)$$

Post composing g_0 with L_t^+ switches the image of g_0 into odd coordinates. Similarly, a homotopy L_t^-, switches image of g_1 into even coordinates. The new maps, labelled g_0, g_1, are connected by $g_t := (1 - t)g_0 + tg_1$. □

Characteristic Classes

Definition 3.7 Let . A characteristic class $c_{n,q}$ is a natural transformation,

$$c_{n,q} : \mathrm{Vect}_n(-) \Rightarrow H^q(-, R)$$

over paracompact spaces. Using the structure in $H^*(-, R)$, We have a graded ring $\Lambda := \bigoplus \Lambda_q$, the *ring of characteristic classes.* [8]

Corollary 3.8 $H^*(G_n(\mathbb{F}^\infty), R) \simeq \Lambda$ *as rings.*

Proof: Let $\alpha \in H^*(G_n(\mathbb{F})^\infty, R)$, an n-vector bundle, $E \to B$. There exists canonical map $g : B \to G_n(\mathbb{F}^\infty)$, by Thm. 3.6. Define $\hat{\alpha} \in \Gamma$ by

$$\hat{\alpha}(E) = g^*(\alpha) \in H^*(B; R)$$

$\hat{\alpha}$ is natural. Thm. 3.6 and naturality shows assignment $\alpha \mapsto \hat{\alpha}$ is bijective. □

We thus compute cohomology of Grassmanians.

Lemma 3.9 *Consider a bundle* $S^{n-1} \to E \xrightarrow{p} B$ *with* E *contractible. If* $n > 1$*, then*

$$H^*(B; R) \cong R[[u]]$$

where u *is as Thm. 3.4.*

Proof: By the LES for fiber homotopy groups, B is $n - 1$ simply connected. Hence, by Thm. 3.4, we have $H^i(B; R) = 0$ for $0 < i < n$ and an isomoprhism $(-) \cup u : H^i(B; R) \to H^{i+n}(B; R)$ for all $i \geq 0$. □

[8] Abusively, an element here is also referred as a characteristic class.

Corollary 3.10

$$H^*(BU(1);R) \cong R[[u]]$$

Proof: We consider the fiber bundle of CW complexes

$$S^1 \to S^\infty \to \mathbb{C}P^\infty = G_1(\mathbb{C})$$

S^∞ is contractible as $\pi_k(S^\infty) = \{0\}, \forall k \geq 0$ and Whitehead theorem applies. □

Proposition 3.11 *The cohomology ring $BU(n)$ is the polynomial ring on generators c_k of degree $2k$ for $1 \leq k \leq n$,*

$$H^*(BU(n);R) \simeq R[[c_1,\ldots,c_n]]$$

Further, $Bi; BU(n_1) \to BU(n_2)$ for $n_1 \leq n_2$, has induced map in cohomology given by

$$(Bi)^*(c_k) = \begin{cases} c_k & \text{for } 1 \leq k \leq n_1 \\ 0 & \text{otherwise} \end{cases}$$

Proof: For $n = 1$, we have $H^*(BU(1)) \simeq R[[c_1]]$. We proceed by induction. Suppose statement is true for $n-1$. Consider canonical map $BU(n-1) \to BU(n)$ and homotopy fiber sequence[9]

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

Thom Gysin and hypothesis implies $H^{2k+1}(BU(n)) = 0$ for all k with SES,

$$0 \to H^{2k-2n}(BU(n)) \to H^{2k}(BU(n)) \xrightarrow{(Bi)^*} H^{2k}(BU(n-1)) \to 0$$

for all k. combining such sequences for all $2k \leq 2n$,

$$0 \to R \xrightarrow{c_n \cup (-)} H^{*\leq 2n}(BU(n)) \xrightarrow{(Bi)^*} H^{2k}(BU(n-1)) \to 0$$

As $H^*(BU(n-1))$ is a free R-algebra, the sequence splits.

$$H^{*\leq 2n}(BU(n)) \simeq (R[[c_1,\ldots,c_n]])_{*\leq 2n}$$

Result follows by induction. □

The c_ks constructed are called the *kth Chern class*. A similar analysis, [Koc96, Prop. 2.3.3], via the spectral sequence $U(n)/U(1)^n \to BU(1)^n \to BU(n)$ gives the splitting principal. [10]

Theorem 3.12 *For $n \in \mathbb{N}$, let $\mu_n : B(U(1)^n) \to BU(n)$ be the canonical map. Then the induced pullback is a monomorphism.*

Theorem 3.13 *Suppose for every vector bundle $E \to B$ of rank n we have assigned $p(E), q(E) \in H^*(B)$, that are natural with respect to the continuous maps. If $p(E) = q(E)$ for every split vector bundle E, then $p(E) = q(E)$ for every vector bundle E.*

Proof: By naturality,$(\mu_n)^*(p(\gamma_n)) = p((\gamma_1)^n) = q((\gamma_1)^n) = (\mu_n)^*(q(\gamma_n))$. From injectivity, Thm. 3.12, p, q coincides on γ_n. By universality of $BU(n)$, Thm. 3.6, result follows. □

[9] In model category, a fibrant resolution

[10] However, a much *simpler* is in [Hat17, Pg. 79], which only uses tools discussed.

3.4 Chern Character Map

We work in $R = \mathbb{Q}$ from now on. As a corollary of Thm. 3.12, Kunneth Theorem and induction, [Koc96, Ch. 2]

Corollary 3.14 *From the induced map $BU(1)^n \to BU(n)$, $H^*(BU(n))$ is isomorphic to the ring of formal power series in degree 2 indeterminates $x_1, \ldots, x_k$ invariant under permutation of x_j. Generators x_j maps to the Euler class $(c_1)_j$ of the canonical line bundle over jth factor.*

Definition 3.15 The *Chern character* is the characteristic class of rank k bundles corresponding to

$$e^{x_1} + e^{x_2} + \cdots + e^{x_k}$$

Lemma 3.16 *Let L and L' be line bundles over M, then $c_1(L \otimes L') = c_1(L) + c_1(L')$.*

Proof: First suppose $L = L' = EU(1)$, $M = BU(1)$ construct L'' over $M \times M$, whose fiber over pair (m, m') is $L_m \otimes L'_{m'}$. By Kunneth,

$$H^2(M \times M) \cong H^2(M) \otimes H^0(M) \oplus H^0(M) \otimes H^2(M).$$

By restriction of L'' to $M \times \{*\}, \{*\} \times M$, and definition of the Kunneth map,

$$c_1(L'') = c_1(L) \otimes 1 + 1 \otimes c_1(L')$$

Restricting to diagonal $M \subseteq M \times M$, we have bundle $L \otimes L'$. By functoriality,

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

□

Proposition 3.17 *Let V, W be complex vector bundles over M. Then*

$$\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W), \quad \mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\,\mathrm{ch}(W)$$

Proof: By Kunneth theorem and Thm. 3.14, the induced map $f_{k,l} : BU(1)^l \times BU(1)^k \to BU(k) \times BU(l)$ in cohomology is injective. It suffices to prove the case for direct sum of line bundles. First equality is clear and by Lem. 3.16.

$$\mathrm{ch}(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1) + c_1(L_2)} = \mathrm{ch}(L_1)\,\mathrm{ch}(L_2)$$

□

Corollary 3.18 *The Chern character gives a homomorphism of rings*

$$\mathrm{ch} : K(M) \to H^{ev}(M)$$

for any compact space M.

The ring structure of K-theory arise from tensor product, [Hat17, Ch.2]. We may also derive it via operator K theory: given a homomorphism of $\mathbb{C}$ algebras, $A \to B$, such that A is commutative and image lies in center of B. Then $B \otimes_{\mathbb{C}} A \to A$ is a ring homomoprhism inducing

$$K(B) \otimes K(A) \to K(B \otimes_{\mathbb{C}} A) \to K(A)$$

making $K(B)$ a module over $K(A)$.

Proposition 3.19 *Let $F(x)$ be a formal power series of x. There is a unique multiplicative class $\mathcal{C}_F$ such that, on line bundles,*

$$\mathcal{C}_F(L) = F(c_1(L)) \in H^*(M)$$

where $c_1(L)$ is the Euler class associated to the bundle $L \to M$.

Proof: The proof is similar as Prop. 3.17. We define for rank n vector bundle, the characteristic class corresponding to $F(x_1) \times \cdots \times F(x_n) \in H^*(BU(n))$. □

Definition 3.20 A characteristic class is *multiplicative* [11] if for any vector bundles V, V',

$$c(V \oplus V') = c(V) \cdot c(V')$$

By Thm. 3.13, they are characterized by the canonical line bundle.

Definition 3.21 The polynomial multiplicative class associated to the formal power series

$$\frac{1}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{720}x^4 + \cdots$$

is the *Todd genus*, denoted Todd.

[11] A multiplicative class $\mathcal{C}$ such that $\mathcal{C}(1) = 1$ where 1 is the trivial line bundle is a *genus*. They are the formal power series in Prop. 3.19 whose zeroth order term is 1.

Chapter 4
Axiomatzation of the Index Theorem

The aim of this chapter is to reformulate Thm. 0.1 axiomatically. In the first section, we first construct explicit K-theory morphisms and classes, due to Higson, [Hig93]. A key tool used is Cor. 2.20. We then conclude with a comparison of Thom homomoprhisms via the Chern character map developed in Chapter. 3. In the second section via a standard local to global argument, we axiomatze the index problem.

We begin this chapter with a new, subtle but very important, definition. A *graded Hilbert space* is a Hilbert space (S, ι) with an orthogonal involution $\iota : S \to S$, such that $S = S^+ \oplus S^-$, where $S^\pm$ are the ± 1 eigenspaces. An EPDO $P : \Gamma(M; S) \to \Gamma(M; E)$ can be turned symmetric via Cor. 1.18. Hence, we may apply functional calculus to P. However, we need a new notion of index, as symmetric EPDOs are of index 0 in the original sense.

Definition 4.1 (Index of symmetric odd graded EPDO D) Let D be a symmetric EPDO on a graded Hilbert bundle $(S, \iota) \to M$, satisfying $D\iota = -\iota D$. With respect to decomposition $S = S_+ \oplus S_-$,

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix}$$

We thus define

$$\mathrm{Ind}(D, \iota) := \mathrm{Ind}(D_+) = \mathrm{Tr}(I|_{\ker(D)})$$

Hence, we will work with symmetric EPDO which are odd-graded, order 1 on a graded Hermitian vector bundle $(S, \iota) \to M$. We will not state conditions when context is clear. We define the topological K theory of locally compact spaces as

$$K^0(X) = K_0(C_0(X))$$

via operator K-theory. We will omit the 0s when context is clear.

4.1 Calculus and K Theory Classes

It is good to review Cor. 2.20, as we will be applying it twice.

We denote $\mathcal{S} := C_0(\mathbb{R})$ with grading $\alpha(f(x)) = f(-x)$, $H = H_0 \oplus H_1$ a graded Hilbert space [1], whose sums are separable infinite dimensional Hilbert spaces.

[1] or graded Hilbert module

Theorem 4.2 *For any (ungraded) C^*-algebra A, there is an isomorphism,*

$$K_0(A) \to [\mathcal{S}, A \otimes \mathcal{K}(H_0 \oplus H_1)]$$

to the homotopy class of graded $*$-homomoprhisms.

One direction of map. Let $[p] - [q] \in K_0(A)$, this induces a a graded $*$-homomoprhism $\mathcal{S} \to A \otimes \mathcal{K}(H_0 \oplus H_1)$ by

$$f \mapsto \begin{pmatrix} pf(0) & 0 \\ 0 & qf(0) \end{pmatrix}$$

identifying $A \otimes \mathcal{K}(H_0 \oplus H_1)$ with $M_2(M_\infty(A))$. [2]

Proof: The construction is given by Quillen, [Qui88].

$$\left\{ \text{Graded maps } \phi \text{ in } \mathcal{S} \to \mathcal{K}(H_0 \oplus H_1) \right\} \rightsquigarrow \left\{ \text{skew unitaries } u_\phi \text{ in } \mathcal{K}(H_0 \oplus H_1)_+ \right\}$$
$$\rightsquigarrow \left\{ \text{Projections in } [p_\phi] - [p_1] \text{ in } K(\mathcal{K}(H_0 \oplus H_1)\ ) \right\}$$

First transformation is by Cayley transform $c : x \mapsto \frac{x+i}{x-i}$, which induces graded morphism

$$C(\mathbb{T}) \xrightarrow{c^*} \mathcal{S}_+ \xrightarrow{\tilde{\phi}} M_2(M_\infty(A))_+ \qquad z \mapsto u_z$$

$C(\mathbb{T})$ is graded by $\gamma(f(w)) = f(\bar{w})$ and generated[3] by inclusion $z : S^1 \to \mathbb{C}$, with $\gamma(z) = z^{-1}$, $zz^* = 1$. So $\alpha(u_\phi) = u_\phi^* = u_\phi^{-1}$. This is a *skew unitary*. [4] which corresponds bijectively to projections

$$u \leftrightsquigarrow \frac{1}{2}(1 + u\varepsilon) =: p_u, \quad \varepsilon := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$p_1 - p_\phi$ lies in $A \otimes \mathcal{K}(H_0 \oplus H_1)$, defining an element

$$[p_1] - [p_\phi] \in K(A \otimes \mathcal{K}(H_0 \oplus H_1)) \simeq K(A)$$

This yields a group isomoprhism, via standard rotation arguments in operator K-theory. □

We may generalize 4.2, [Hig14, Prop. 3.32]

Proposition 4.3 *A graded $*$-homomoprhism $\mathcal{S} \otimes A \to B \otimes \mathcal{K}$ determines a K-theory map*

$$\phi_* : K(A) \to K(B)$$

such that:

1. *$\phi \mapsto \phi_*$ is functorial with respect to each argument.*

[2] The grading is given by diagonal matrices even and off diagonal are odd.
[3] By Stone-Weierstrass.
[4] Skew unitaries v in graded unital C^* algebra, C, with grading β, are unitaries such that $\beta(v) = v^*$.

2. ϕ_* *depends only on homotopy class*

As an application, we construct a K-theory class that yields $\mathrm{Ind}(D, \iota)$.

Corollary 4.4 *The element* $[\phi_D] \in K(\mathbb{C}) \simeq \mathbb{Z}$ *is the Frehdolm index of the operator of a symmetric odd graded EPDO* D*, where*

$$\phi_D : \mathcal{S} \to \mathcal{K}(H)$$

is defined via functional calculus.

Proof: We define for each $s \in (0,1]$ the following map via functional calculus,

$$\phi_{s^{-1}D} : f \mapsto f(s^{-1}D) \in [\mathcal{S}, \mathcal{K}(L^2(M))]$$

As $\lim_s f(s^{-1}x) = \delta_0(x)$, we have a homotopy to the projection on 0 eigenspace. So

$$[P|_{H_+}] - [P|_{H_-}]$$

is our K-theory class in $K(\mathbb{C}) \simeq K_0(\mathcal{K}(H_0 \oplus H_1))$. Taking traces yield

$$\dim(\ker(D) \cap H_+) - \dim(\ker(D) \cap H_-) = \mathrm{Ind}(D, \iota).$$

□

K-Theory Class from Symbol

We construct an element in $[\sigma_D] \in K(T^*M)$ from σ_D symbol.

Definition 4.5 Let S be a graded Hermitian vector bundle over locally compact Z, $c : S \to S$ a self-adjoint, odd endomorphism. $c : S \to S$ is *elliptic* if the operator norm of

$$(I + c(z)^2)^{-1} : S_z \to S_z$$

vanish at infinity.

The symbol, regarded as $\sigma_D : \pi^*S \to \pi^*S$, of a symmetric EPDO, is an elliptic morphism. Ellipticity shows $(\sigma_D)_\xi$ is bounded below on the complement of an neighborhood of zero section in T^*M. From homogenity, $(\sigma_D)_{t\xi} = t(\sigma_D)_\xi$, we obtain the vanishing condition.

Definition 4.6 We define $c \in K(Z)$ the *difference class* of elliptic endomorphism $c : S \to S$:

By functional calculus, there is a graded $*$-homomorphism with in the compact orators of Hiblert module $C_0(Z,S)_{C_0(Z)}$.

$$\phi_c : \mathcal{S} \to \mathcal{K}_{C_0(Z)}(C_0(Z,S))$$
$$f \mapsto f(c)$$

This is a K-theory class, by Cor. 2.20.

$$c \in K(Z) \simeq K_0(\mathcal{K}_{C_0(Z)}(C_0(Z,S))).$$

The *symbol class*[5] of D is the difference class $\sigma_D \in K^0(T^*M)$.

[5] In Thm. 4.2 the symbol class yields unitary $u_D := (\sigma_D + iI)(\sigma_D - iI)^{-1}$

Thom Homomorphism of K-Theory

We use functional calculus to extend the Thom homomoprhism of K-theory over compact spaces. Let $\Lambda^* V$ be the exterior algebra of vector bundle $\pi : V \to X$, $\mathbb{Z}/2\mathbb{Z}$ graded by even and odd degrees. Define

$$b : \pi^* \Lambda^* V \to \pi^* \Lambda^* V$$

$$b(v)s = v \wedge s + v \lrcorner s$$

Locally, $b(v)^2 = ||v||^2 \cdot I$, so b is an elliptic element as Def. 4.5. Functional calculus induces graded $*$-homomorphism

$$\phi : \mathcal{S} \otimes C_0(X) \to \mathcal{K}_{C_0(Z)}\Big(C_0(V, \Lambda^* V)\Big)$$

$$\phi(f \otimes h)(v) = f(b(v))h(\pi(v))$$

The image identifies with the algebra of compact operators on the Hilbert module $C_0(V, \Lambda^* V)_{C_0(V)}$.

By general formula Thm. 4.3, and Cor. 2.20,

$$\phi : K_0(C_0(X)) \to K_0(C_0(V))$$

Definition 4.7 The homomorphism $\phi : K(X) \to K(V)$ is also called the *Thom homomorphism*.

Definition 4.8 If $V \to X$ is complex Hermitian bundle over compact X, the *Thom element (class)* $b_V \in K(V)$ is difference class as given in Def. 4.6.

Over compact space X, the homomoprhism coincides with

$$\phi' : K(X) \to K(V), \quad x \mapsto \pi^*(x) \cdot b_V$$

By Bott periodicity and a Cayley transform, [Fra18, Prop. 15],

Theorem 4.9 *Let V be a finite dimensional complex Hermitian vector space. The abelian group $K(V)$ is freely generate by the Thom element (Bott element.)*

Comparison of the Thom Homomorphisms

We may extend the definition of Chern character from compact to locally compact spaces by one point compactification X^+.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(X) & \longrightarrow & K(X^+) & \longrightarrow & K(*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \\ 0 & \longrightarrow & H^{ev}_c(X) & \longrightarrow & H^{ev}(X^+) & \longrightarrow & H^{ev}(*) & \longrightarrow & 0 \end{array}$$

Proposition 4.10 *Let V be a k-dimensional complex vector bundle over closed manifold M. Define $\tau(V) \in H^{ev}(M)$ by the formula*

$$\mathrm{ch}(b_V) = \tau(V) \cdot u_V$$

where b_V is the K-theory Thom class, and $u_V \in H^{ev}(V)$ is the cohomology Thom class. Then $\tau(V)$ is a multiplicative characteristic class.

Proof: Functoriality is clear. We can view V_2 as a vector bundle over the total space of V_1 by pulling back to V_1. Then compositions of Thom homomoprhisms

$$K(M) \xrightarrow{\phi_v} K(V_1) \xrightarrow{\phi_{\pi^* V_2}} K(V_1 \oplus V_2)$$

$$H^{ev}(M) \xrightarrow{\psi_V} H^{ev}(V_1) \xrightarrow{\psi_{\pi^* V_2}} H^{ev}(V_1 \oplus V_2)$$

are equal[6] to the Thom homomoprhisms in K-theory for the complex vector $V_1 \oplus V_2$ over M. For K-theory, this is seen by using the $*$-homomorphism representation,[Hig14, Lemma 5.15]. From functoriality, ring morphism property, and that Thom homomorphism in cohomology is a $H^*(M)$ module homomorphism.

$$\mathrm{ch}(\phi_{V_1 \oplus V_2}(x)) = \tau(V_1)\tau(V_2)\psi_{V_1 \oplus V_2}(x)$$

□

By Thm. 3.19, τ is given by a formal power series, evaluated at canonical bundle $EU(1) \to BU(1)$. By an induction on filtration of $\mathbb{C}P^\infty$, we have, [Hig14, Thm. 5.16].

Lemma 4.11 *τ is associated topower series $(1 - e^x)/x$. As a special case of the proof, for complex vector space W of dimension k,*

$$\int_W \mathrm{ch}(b_W) = (-1)^k$$

4.2 Axiomitization of the Index Map

We think of Thom homomoprhism as a wrong way map. Another is the inclusion of open sets $X \hookrightarrow Y$, inducing

$$i_! : K(X) = K_0(C_0(X)) \to K_0(C_0(Y)) = K(Y)$$

Chern character is not functorial in the wrong way - and its defect measured by τ. We now axiomatize the index theory problem, [Hig93].

Theorem 4.12 *Assume that to every manifold M, there is associated a homomorphism $\mathrm{Ind}_{a,M}$: $K(T^*M) \to \mathbb{Z}$ with the following properties:*

(i) If $i : M_1 \hookrightarrow M_2$ is an embedded as an open subset of M_2, then the diagram

$$\begin{array}{ccc} K(T^*M_1) & \xrightarrow{\mathrm{Ind}_{a,M_1}} & K(*) \\ \downarrow{\scriptstyle i_!} & & \downarrow \\ K(T^*M_2) & \xrightarrow{\mathrm{Ind}_{a,M_2}} & K(*) \end{array}$$

commutes, where $i_!$ is the wrong way open inclusion from embedding.

[6] via the canonical isomorphism $\pi^* V_2 \simeq V_1 \oplus V_2$

(ii) If V is a real vector bundle of dimension k over M, and if $\phi : K(T^*M) \to K(T^*V)$ denotes the Thom homomoprhism, then the diagram

$$\begin{array}{ccc} K(T^*M) & \xrightarrow{\mathrm{Ind}_{a,M}} & K(*) \\ \downarrow & & \downarrow \\ K(T^*V) & \xrightarrow{\mathrm{Ind}_{a,V}} & K(*) \end{array}$$

commutes.

(iii) If $b \in K(T^*\mathbb{R}^n)$ is the Bott element, then $\mathrm{Ind}_{\mathbb{R}^n}(b) = 1$.

Then [7]

$$\mathrm{Ind}_{a,M}(x) = (-1)^{\dim(M)} \int_{T^*M} \mathrm{Todd}(TM \otimes \mathbb{C}) \,\mathrm{ch}(x)$$

for every M and $x \in K(T^*M)$.

We explain the orientations. For (ii), by choice of an Euclidean metric, identify $TM \simeq T^*M$, and $TV \simeq T^*V$. The pullback $\pi : TM \to M$ induces isomoprhism

$$TV \simeq \pi^*(V \oplus V)$$

and we identify $V \oplus V = V \otimes \mathbb{C}$, giving orientation fo the other bundles. For (iii), we orient T^*M by choosing a local coordinates $\{x_i\}_1^n$, with dual, $\{dx^i\}_1^n$, so that $\{x_i, dx^i\}_1^n$ is an orientation on T^*M.

Proof: We start with $\mathbb{R}^n$ case. By normalization axiom, when $x = b$, LHS is $1 \in K(*)$. As the Todd genus is a *multiplicative* characteristic class and its evaluation is 1 on trivial bundles, we are reduced to Lem. 4.11. By Bott Periodicity, Thm. 2.33, b generates $K(T^*\mathbb{R}^n)$. We are thus done when $M = \mathbb{R}^n$.

By functoriality of axiom (i), the statement holds for all open subsets of $\mathbb{R}^n$. Now take a tubular embedding, $i : M \hookrightarrow \mathbb{R}^n$, normal bundle $N_iV \to M$, where N_iM is homeomorphic to an open subset of $\mathbb{R}^n$. By axiom (i) again, the theorem holds for T^*N_iM. We apply axiom (ii). Commutativity implies

$$\mathrm{Ind}_{a,M}(x) = \mathrm{Ind}_{a,V}(\phi(x)) = (-1)^{n+k} \int_{T^*N_iM} \mathrm{ch}(\phi(x))$$

By Prop, 4.11 and Thm. 4.10

$$\mathrm{Ind}_{a,M}(x) = (-1)^{n+k} \int_{T^*M} \tau(V \otimes \mathbb{C}) \,\mathrm{ch}(x).$$

As $N_iM \oplus TM \simeq \mathbb{R}^{n+k}$ is a trivial bundle, using multiplicative property of τ, Todd, and Lem. 4.11, giving

$$\tau(V \otimes \mathbb{C}) = (-1)^k \,\mathrm{Todd}(TM \otimes \mathbb{C})$$

[7]Note that as both elements lie in even cohomology the order of product does not matter in integral.

□

We connect to Sec. 3.1. Axiom (iii) says $\mathrm{Ind}_{\mathbb{R}^n} : K(T^*\mathbb{R}^n) \simeq K(T\mathbb{R}^n) \to K(*)$ is the inverse of suspension

$$K(*) \to K(T\mathbb{R}^n) \simeq K(\mathbb{C}^n), x \mapsto x \cdot b_V$$

induced by Bott element, Thm. 4.9. We proceed with fiber integration.

Definition 4.13 The *topological index* $\mathrm{Ind}_{t,i}$ of a compact smooth manifold M, with a choice of tubular embedding $i : M \hookrightarrow \mathbb{R}^n$, is given by

$$\begin{array}{ccc}
K(T^*N_iM) & \xrightarrow{i_!} & K(T^*\mathbb{R}^n) \simeq K(\Sigma^n \wedge \{*\}_+) \\
\uparrow{\scriptstyle Thom,\simeq} & & \downarrow{\scriptstyle susp,\simeq} \\
K(T^*M) & \overset{\mathrm{Ind}_{t,i}}{\dashrightarrow} & K(*) \simeq \mathbb{Z}
\end{array}$$

On the other hand, we want the analytic index map to satisfy Thm. 4.12 and

Definition 4.14 (The analytic map) For each manifold M, there is a homomorphism

$$\mathrm{Ind}_{a,M} : K(T^*M) \to K(*)$$

such that if M is compact, and if $\sigma_D \in K^0(T^*M)$ is the symbol class of an elliptic operator D on M, then $\mathrm{Ind}_a(\sigma_D) = \mathrm{Ind}(D)$ in $K(*) \simeq \mathbb{Z}$.

The problem therefore becomes[8]

$$\mathrm{Ind}_a = \mathrm{Ind}_{t,i}$$

[8]We then have independence of embedding!

Chapter 5
Groupoids and $*$-Asymptotic Morphisms

In this chapter, we follow the proof of Higson's, [Hig14], which avoids the use of pseudodifferential operators. This is done via asymptotic $*$-morphisms. The background developed here would ultimately help one understand more the second proof in Chapter 6.

We begin by quickly reviewing standard groupoids. In the second section we describe functorial properties of asymptotic $*$-morphisms, and in the last, we construct the key groupoid algebra which allows us to prove the symbol class is mapped to the topological index, Thm. 5.12, via the index map in the end of Chapter. 4.

The prerequisites of this chapter are the definition of groupoids, Haar systems and groupoid algebras, [Ren80]. A groupoid is denoted as $G \rightrightarrows G^{(0)}$. We let r, s be the source and range maps, and identify $G^{(0)}$ in G via the unit map. $G_A := s^{-1}(A), G^B = r^{-1}(B)$ and $G_A^B = s^{-1}(A) \cap r^{-1}(B)$. A groupoid is *smooth* if G and $G^{(0)}$ are smooth manifolds, all morphisms are smooth, unit map is an embedding, and r, s are submersions. We will only be working with smooth groupoids.

5.1 Standard groupoids

We list a few important groupoids.

Definition 5.1 If $G \rightrightarrows G^{(0)}$, $\varphi : X \to G^{(0)}$ is an open surjective map, the *pullback groupoid*, $\varphi^*(G) \rightrightarrows X$ given by

$$\begin{aligned}\varphi^*(G) = \{(x, \gamma, y) \in X \times G \times X \,:\, \varphi(x) = r(\gamma), \varphi(y) = s(\gamma)\\ s(x, \gamma, y) = y\\ r(x, \gamma, y) = x\\ (x, \gamma_1, y) \cdot (y, \gamma_2, z) = (x, \gamma_1 \cdot \gamma_2, z)\\ (x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)\end{aligned}$$

Definition 5.2 A groupoid G is called a *deformation groupoid* if

$$G = G_1 \times \{0\} \cup G_2 \times (0, 1] \rightrightarrows G^{(0)} = M \times (0, 1]$$

where G_1 and G_2 are groupoids with unit space M.

Definition 5.3 The *tangent groupoid* is the gluing of tangent bundle with pair groupoid,

$$\mathcal{G}_M^t := TM \times \{0\} \sqcup M \times M \times (0,1] \rightrightarrows M \times [0,1]$$

We may assign a smooth structure on $\mathcal{G}_M^t$.

Definition 5.4 Two groupoids $G \rightrightarrows G^{(0)}, H \rightrightarrows H^{(0)}$ are *Morita equivalent* if there exists groupoid $P \rightrightarrows G^{(0)} \sqcup H^{(0)}$ such that

- $P_{G^{(0)}}^{G^{(0)}} = G, P_{H^{(0)}}^{H^{(0)}} = H$
- For any $\gamma \in P$ exists $\eta \in P_{G^{(0)}}^{H^{(0)}} \cup P_{H^{(0)}}^{G^{(0)}}$ such that (γ, η) is composable.

A groupoid $G \rightrightarrows G^{(0)}$ can be viewed as a family of manifolds,

$$G_x = \{\gamma \in G \,:\, s(\gamma) = x\}$$

parameterized by $x \in G^{(0)}$. For smooth groupoid, a Haar system always exists.

Definition 5.5 Given $x \in G^{(0)}$, $f \in C_c(G)$, $\xi \in L^2(G_x, \nu_x)$, $\gamma \in G_x$ we set

$$\pi_x(f)(\xi)(\gamma) = \int_{\eta \in G_x} f(\gamma\eta^{-1})\xi(\eta)\, d\nu_x(\eta)$$

π_x is a $*$-representation of $C_c(G)$ on the Hilbert space $L^2(G_x, \nu_x)$. The inequality

$$||\pi_x(f)|| \leq ||f||_1$$

holds. The *reduced norm* on $C_c(G)$ is

$$||f||_r = \sup_{x \in G^{(0)}} \{||\pi_x(f)||\}$$

which defines a C^*-norm. The *reduced* C^* algebra $C_r(G, \nu)$ is the completion of A as C^* algebra with respect to $|| \cdot ||_r$.

When G is smooth, the reduced and maximal C^*-algebras does not depend up to isomoprhism of Haar system ν. All groupoids we use have full and reduced norm equal, [Con95]. [1] A standard example of groupoid algebra is

$$C^*(TM) \simeq C_0(T^*M)$$

To see this isomoprhism, equip TM with a Riemannian structure. We let $f \in C_c(TM)$ and define a morphism into $C_0(T^*M)$ by

$$(x, w) \in T^*M, \hat{f}(x, w) = (2\pi)^{-\frac{n}{2}} \int_{X \in T_xM} e^{-iw(X)} f(X)\, dX$$

Via Fourier theory, this is $*$-isometric.

[1] This is true for a large class of groupoids, the amenable groupoids.

Proposition 5.6 *Let X be smooth manifold, μ a positive regular borel measure on X, then*

$$C^*(X \times X) = C_r^*(X \times X) \simeq \mathcal{K}(L^2(X, \mu))$$

Proof: The $*$-representation

$$\pi : C_c(X \times X) \to \mathcal{L}(L^2(X, \mu)) \, : \, \pi(f)(\xi)(x) = \int_{z \in X} f(x, z)\xi(z) \, d\mu(z)$$

is in fact $*$-isometric. Further, the image is a dense subset of Hilbert Schimdt operators on $L^2(X, \mu)$, of which are dense in compact operators, [Gar14, Ch. 6]. □

5.2 Asymptotic $*$- Morphisms

We will use Higson's, [Hig93], asymptotic morphisms to construct index map.

Definition 5.7 Let A and B be C^* algebra. An *asymptotic morphism* from A to B is a family of functions $\phi_t : A \to B$, $t \in [1, \infty)$ such that

- For each $a \in A$, the map $t \mapsto \phi_t(a) \in B$ is coutinuous and bounded.
- For all $a, a_1, a_2 \in A$, $\lambda_1, \lambda_2 \in \mathbb{C}$

$$\lim_{t \to \infty} \begin{Bmatrix} \phi_t(a_1 a_2) - \phi_t(a_1)\phi_t(a_2) \\ \phi_t(\lambda_1 a_1 + \lambda_2 a_2) - \lambda_1 \phi_t(a_1) - \lambda_2 \phi_t(a_2) \\ \phi_t(a^*) - \phi_t(a)^* \end{Bmatrix} = 0$$

Theorem 5.8 *An asymptotic morphism $\phi_t : A \to B$ determines a K-theory map $\phi_* : K(A) \to K(B)$ with the following properties:*

1. *The correspondence $\phi \mapsto \phi_*$ is functorial with respect to composition with $*$-homomorphisms, $A_1 \to A$ and $B \to B_1$.*
2. *If each ϕ_t is actually a $*$-homomorphism, then $\phi_* : K(A) \to K(B)$ is the map induced by ϕ_1.*

Proof: We construct the map ϕ_* - that it satisfies the properties is a check. Let $\mathcal{A}(B)$ denote the space of continuous bounded functions from $[1, \infty)$ to B. This induces an exact sequence

$$0 \to \mathcal{J}(B) \to \mathcal{A}(B) \to \mathcal{Q}(B) \to 0$$

where $\mathcal{J}(B)$ is the algebra of functions which vanish at infinity, and is contractible. By Cor. 2.34,

$$K_0(\mathcal{A}(B)) \xrightarrow{\simeq} K_0(\mathcal{Q}(B))$$

An asymptotic morphism ϕ_t induces a map $\tilde{\phi} : A \to \mathcal{Q}(B)$, by $a \mapsto t \mapsto [\phi_t(a)]$. Hence,

$$\begin{array}{ccc} K_0(A) & \overset{\phi_*}{\dashrightarrow} & K_0(B) \\ \downarrow{\scriptstyle \tilde{\phi}_*} & & \uparrow{\scriptstyle ev_1} \\ K_0(\mathcal{Q}(B)) & \underset{\simeq}{\longleftarrow} & K(\mathcal{A}(B)) \end{array}$$

where ev_1 is evaluation map at 1. $\square$

We describe here, [Hig93, Pg. 3] the action of ϕ on p, a projection, in unital A. As

$$\lim_t ||\phi_t(p) - \phi_t(p)^2|| = 0$$

Functional calculus implies exists a continuous family of projections $q_t \in B$ such that

$$||\phi_t(p) - q_t|| \to 0$$

We then define

$$\phi[p] := [q_1]$$

5.3 Tangent Groupoid

We construct Ind_a. Let M be a compact manifold with a fixed positive regular Borel measure ν on M. Consider its tangent groupoid $\mathcal{G}^t_M$. Then μ determines a family of translation invariant measure μ_m on $T_m M$. We define smooth measures on the fibers of $\mathcal{G}^t_M$ by

$$\begin{cases} \mu_{m,0} = \mu_m & \text{on} \quad \mathcal{G}^t_{M\,(m,0)} \simeq T_m M \\ \mu_{(m,t)} = t^{-n}\mu & \text{on} \quad \mathcal{G}^t_{M\,(m,t)} \simeq M \end{cases}$$

This defines a smooth right invariant system, [Hig14, Lem. 7.23], with two $*$-morphisms

$$\varepsilon_0 : C^*(\mathcal{G}^t_M) \to C_0(T^*M)$$

by restriction to zero section, and for $t \neq 0$, the restricted representation, [2]

$$\varepsilon_t : C^*(\mathcal{G}^t_M) \to \mathcal{K}(L^2(M))$$

using Prop. 5.6.

By choosing a not necessarily concintuos section σ of surjection ε_0,

$$C^*(\mathcal{G}^t_M) \longrightarrow C_0(T^*M), \qquad \sigma : C_0(T^*M) \to C^*(\mathcal{G}^t_M)$$

we can define the following asymptotic $*$-homomorphism, [Hig93]

Proposition 5.9 *The family of maps, $\alpha_t : C_0(T^*M) \to \mathcal{K}(L^2(M))$*

$$\begin{array}{ccc} C_0(T^*M) & \xrightarrow{\ \alpha_t\ } & \mathcal{K}(L^2(M)) \\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle\varepsilon_{t^{-1}}} & \\ & C^*(\mathcal{G}^t_M) & \end{array}$$

$$\alpha_t(h) = \varepsilon_{t^{-1}}(\sigma(h)), \quad t \in [1, \infty), h \in C_0(T^*M)$$

[2] For each t, lies in operators on $L^2(M, t^{-n}\mu)$, these Hilbert spaces are unitarily equivalent by scalaing $t^{n/2}$

This will be our index homomorphism once we relate symmetric EPDO D to groupoids. We revisit the symbol operator. If $D = \sum a_j \partial_j + b$ in local coordinates, acting on $\Gamma(M, S)$, then σ_D maps $\xi \in T^*M$, to the endomorphism

$$\sum a_j(m)\xi^j : S_\xi \to S_\xi$$

where S_ξ denotes a fiber. Thus, this determines translation invariant operator on each $m \in M$

$$D_m : \Gamma(T_m M; T_m M \times S_m) \to \Gamma(T_m M; T_m M \times S_m)$$

where $D_m = \sum a_j(m)\partial_j$. The operators D_m are the *model operators.* [3]

We associate to D a family of smooth varying PDOs on the fibers.

$$\begin{cases} D_{m,0} = D_m & \text{on } \mathcal{G}^t_{M(m,0)} \simeq T_m M \\ D_{m,t} = tD & \text{on } \mathcal{G}^t_{M(m,t)} \simeq M \end{cases}$$

Utilizing the first order condition, we have:

Theorem 5.10 *[Hig14, Thm. 7.33], [Che73] Let $D = \{D_x\}$ be a smooth right translation invariant family of symmetric EPDOs on the fibers G_x of G with compact base space, there is a $*$-homomoprhism*

$$\tilde{\phi}_D = C_0(\mathbb{R}) \to C^*_r(G, \nu)$$

with the property that if $x \in G$, for all regular representation $\pi_x : C_c(G) \to L^2(G_x, \nu_x)$, as Def. 5.5,

$$\pi_x(\tilde{\phi}_D(f)) = f(D_x) : L^2(G_x) \to L^2(G_x)$$

for every $f \in C_0(\mathbb{R})$.

Consider when D acts on the *trivial one dimensional complex manifold*.

Proposition 5.11 *Let D be a first order elliptic operator on the one dimensional trivial bundle of a closed manifold M. Let $f \in C_0(\mathbb{R})$. The index asymptotic morphism, $\alpha_t : C_0(T^*M) \to \mathcal{K}(L^2(M))$ maps to $f(\sigma_D) \in C_0(T^*M)$ to the family $\alpha_t(\sigma_D) = f(t^{-1}D) \in \mathcal{K}(L^2(M))$.*

Proof: $\sigma : C_0(T^*M) \to C^*(\mathcal{G}^t_M)$ only has to be a *set theoretic* section. The symbol σ_D acts on the fibers, $\mathbb{C}$, so $f(\sigma_D) \in C_0(T^*M)$. We define the section on this image by

$$\sigma(f(\sigma_D)) := \tilde{\phi}_D(f)$$

using Thm. 5.10. We define arbitrary section on the complement.[4] The index map is given by Prop. 5.9. □

Theorem 5.12 *Let D be a first order symmetric odd graded EPDO on a closed manifold M, the index map $\alpha : K(T^*M) \to K(*)$ from Prop. 5.11, maps $[\sigma_D]$ to $\mathrm{Ind}(D, \iota)$.*

[3] In otherwords, D may be regarded as a EPDOs with constant coefficients locally.

[4] Since we need not a continuous section

Proof: Via Thm. 4.2 we have correspondence

$$[f \mapsto f(t^{-1}D)] \in [\mathcal{S}, \mathcal{K}(L^2(M))] \rightsquigarrow u_t := (t^{-1}D + iI)(t^{-1}D - iI)^{-1}$$
$$[f \mapsto f(\sigma_D)] \in [\mathcal{S}, C_0(T^*M)] \rightsquigarrow u_D := (\sigma_D + iI)(\sigma_D - iI)^{-1}$$

Prop. 5.11 implies

$$\lim_t ||\alpha_t(u_D) - u_t|| \to 0$$

$$\alpha_*([\sigma_D]) = \alpha_*([p_1] - [p_{\sigma_D}]) = [p_1] - [p_{u_1}]$$

This is Prop. 4.4. □

The argument generalizes verbatim when D acts on a graded Hermitian vector bundle S over M, with groupoid algebra $C_c^\infty(G, \mathrm{End}(S))$.

The last step

We will end our discussion of Higson's proof here. *We are done with order* 1 *case provided compatibility with Thom homomorphism*, this is done analyzing asymptotic $*$-morphisms, [Hig14, Thm. 8.12].

Chapter 6
The Kasparov Product

This chapter provides a second proof of Atiyah-Singer, due to [DLN06]. This proof proves the general case and utilizes all the tools developed in the preivous chapters. We omit a step in justifying why Def. 6.20 is the analytic map, and refer to the original source. This requires notions of pseudodifferential operators, which we do not have space to develop.

The proof in [DLN06] uses a bivariant K-theory, KK-theory. The presentation here wishes to highlight the formal properties of KK-theory and the final proof of the Atiyah Singer theorem. Hence, in the first two sections, after defining Kasparov modules, and discussing some axiomatic properties, we look into universal characterization, [Hig87].

In the grand scheme, Higson's proof in Chapter 5, motivates E-theory, [Con95], a category whose objects are C^* algebras but with asymptotic C^* homomorphisms as hom-sets. It can be regarded as the universal half exact localization of C^* Alg. KK-theory, on the other hand, is the universal split exact localization. We will see this in the second section.

Hopefully, after exposure to the two proofs of Atiyah-Singer index theorem, the reader can start appreciating the role of deformation groupoids, and part of the noncommutative language of index theory.

6.1 Kasparov Modules

Definition 6.1 A *Kasparov A-B-module* is given by a triple $x = (\mathcal{E}, \pi, \mathcal{F})$ where $\mathcal{E} = \mathcal{E}^0 \oplus \mathcal{E}^1$ is $\mathbb{Z}/2\mathbb{Z}$-graded and countably generated Hilbert B-module. $\pi : A \to \mathrm{Mor}(\mathcal{E})$ is a $*$-morphism of degree 0 with respect to the grading, and $F \in \mathrm{Mor}(\mathcal{E})$ is of degree 1. For all[1] $a \in A$,

1. $\pi(a)(F^2 - 1) \in \mathcal{K}(\mathcal{E})$
2. $[\pi(a), F] \in \mathcal{K}(\mathcal{E})$.

we denote the set of Kasparov A-B modules by $E(A, B)$.

Definition 6.2 A Kasparov A-B module $x = (\mathcal{E}, \pi, \mathcal{F})$ is *degenerate* if for all $a \in A$, $\pi(a)(F^2 - 1) = 0$ and $[\pi(a), F] = 0$.

As operator K-theory, we have the notions of sum, pullback, inner[2] and outer tensor products, [JT91]

[1] Contrary to original, we omit condition $\pi(a)(F - F^*) \in \mathcal{K}(\mathcal{E})$, which was observed unnecessary, [Ska91].
[2] Care must be taken for induced morphisms. This is where Connes introduced the notion of connections.

We also have homotopy, with evaluation $*$-morphisms,

$$ev_t : IB \to B, \quad f \mapsto f(t), \quad IB := C([0,1], B)$$

Definition 6.3 Two Kasparov A-B modules $(\mathcal{E}_i, \phi_i, \mathcal{F}_i)$, $i = 0, 1$ are *homotopic* if there is a Kasparov A-IB module, $(\mathcal{E}, \phi, \mathcal{F}) \in E(A, IB)$ with $ev_{0*}(\mathcal{E}) \simeq \mathcal{E}$ and $ev_{1*}(\mathcal{E}) \simeq \mathcal{F}$ as Kasparov A-B modules. The set of homotopy classes is denoted $KK(A, B)$

In particular, degenerate elements of $E(A, B)$ are homotopic to $(0, 0, 0)$ and as operator K-theory,

Proposition 6.4 *Sum of Kasparov modules give $KK(A, B)$ a structure of an abelian group.*

We end with two useful examples.

Proposition 6.5 *Each $*$-morphism $f : A \to B$ defines an element, by $[f] \in KK(A, B)$. We set $1_A := [id_A] \in KK(A, A)$.*

Proof: Since $\mathcal{K}_B(B) \simeq B$, f induces a representation on B as a Hilbert B-module, hence

$$[f] := [(B, f, 0)]$$

is a A-B Kasparov module. □

Proposition 6.6 $KK(\mathbb{C}, B) \cong K_0(B)$.

Proof: We sketch the unital case. A finitely generated projective $\mathbb{Z}/2$ graded B module $\mathcal{E}$ is submodule of $B^N \oplus B^N$ for some N. Hence, has the structure of Hilbert B module. Further [DL08, Prop. 4.26] $Id_{\mathcal{E}}$ is a compact operator (we require unital B), so

$$(\mathcal{E}, \iota, 0) \in E(\mathbb{C}, B)$$

where ι is the scaling representation of $\mathbb{C}$ on $\mathcal{E}$. This yields an group morphism $K_0(B) \to KK(\mathbb{C}, B)$. An inverse construction can be given. □

6.2 Universal Characterization of KK-Theory

Success of Kasparov's work, [Kas80], lies in the *Kasparov product*, motivated by Fredholm operators. Higson, [Hig87], exploited the Cuntz picture, seeing $KK(A, B)$, as *quasihomomorphisms*, [Cun83]. KK is the universal

Definition 6.7 $F : C^* \,\mathrm{Alg} \to \mathrm{Ab}$

1. F is homotopy functor.
2. For every separable C^* algebra B, the homomoprhism $e_* : F(B) \to F(\mathcal{K} \otimes B)$ is an isomoprhism.
3. F preserves split exact sequences:

   $$0 \longrightarrow I \longrightarrow A \rightleftarrows A/I \longrightarrow 0$$

   is a split exact sequence then so is

   $$0 \longrightarrow F(I) \longrightarrow F(A) \rightleftarrows F(A/I) \longrightarrow 0$$

Characterization begins with a Yonneda-like observation, [Hig87, Thm. 3.7],

Theorem 6.8 *There is a natural bijection*

$$\mathrm{Hom}(KK(A,-), F-) \cong FA$$

where Hom *denotes the set of natural transformations, i.e. each* $x \in A$ *yields unique transformation* τ, $\tau(1_A) = x$.

Kasparov proved the existence of the pairing, [Kas80]

Theorem 6.9 *There exists a bilinear pairing* $KK(A,B) \times KK(B,C) \to KK(A,C)$ *denoted* $x, y \mapsto x \otimes_B y$ *such that*

1. *If* $f : A' \to A$, $f^*(x \otimes_B y) = f^*(x) \otimes_B y$.
2. *If* $g : B \to B'$ *then* $g_*(x) \otimes_{B'} z = x \otimes_B g^*(z)$ *where* $z \in KK(B', C)$.
3. *If* $h : C \to C'$, *then* $h^*(x \otimes_B y) = x \otimes_B h_*(y)$
4. $1_A \otimes_A x = x \otimes_B 1_B = x$.

Applying this pairing, all axioms of Def. 6.7 for $KK(A,-)$ are satisfied. Importantly, from the construction Prop. 6.5, we have the compatibility

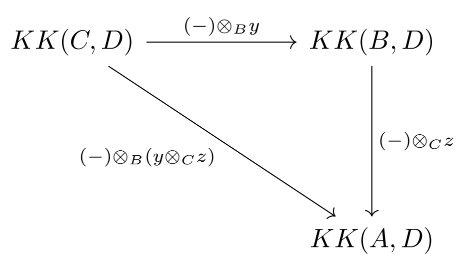
Corollary 6.10 *Let* $f : C \to A, g : B \to E, h : D \to C,\ x \in KK(A,D), z \in KK(C,B)$, *then*

$$f^*(x) = [f] \otimes_A x, g^*(z) = z \otimes_B [g] \text{ and } [f \circ h] = [h] \otimes_C [f]$$

We prove associativity.

Theorem 6.11 *The Kasparov product is associative.*

Proof: Let $x \in KK(A,B), y \in KK(B,C)$. We have two natural transformations



satisfying $1_C \mapsto x \otimes_B y$ using properties of 6.9. Hence, by 6.8, they coincide. □

Definition 6.12 Let kk be a category whose objects are separable C^* algebras and morphisms are $KK(A,B)$. Law of composition $y \circ x$ is given by $x \otimes_B y$.

Corollary 6.13 *The Kasparov product, is the* Hom *funtor of an additive category* kk,

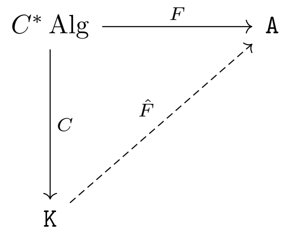
$$KK(-,-) : C^*\,\mathrm{Alg} \times C^*\,\mathrm{Alg} \to C^*\,\mathrm{Alg}$$

There is a canonical functor

$$i : C^* \operatorname{Alg} \to \mathtt{kk}, \quad A \mapsto A, \quad i(f) = f_*(1_A)$$

where $1_A \in KK(A, A)$. We come to the universal characterization of KK-theory.

Theorem 6.14 *The canonical functor $i : C^* \operatorname{Alg} \to \mathtt{kk}$, in 6.13, is universal with respect to all functors $F : C^* \operatorname{Alg} \to \mathtt{A}$ be satisfying the axioms of definition 6.7. (except that codomain is $\mathtt{kk}$.)*



Proof: Axioms imply $\mathtt{A}(X, F(-)) : C^* \operatorname{Alg} \to \operatorname{Ab}$ is a homotopy invariant, stable, and split exact functor. On objects, $\hat{F}(A) = F(A)$. For $x \in \mathtt{kk}(A, B) = KK(A, B)$, we define $\hat{F}(x)$ to be image of x under the the natural transformation corresponding to $1_{F(A)} \in \mathtt{A}(F(A), F(A))$. This yields uniqueness. It is also functorial, [Hig87, Thm. 4.2]. □

Deformation Groupoids

There are two important results which allows us to construct KK elements. [HS83, Cor. 9].

Theorem 6.15 *$G \rightrightarrows G^{(0)}$ has Haar system ν. An open subset $U \subseteq G^{(0)}$ is saturated if $U = s(r^{-1}(U))$.*

The set $F = G^{(0)} \setminus U$ is closed saturated. The Haar system ν restricted to $G|_U := G_U^U$ and $G|_F := G_F^F$ induces the exact sequence

$$0 \to C^*(G|_U) \xrightarrow{i} C^*(G) \xrightarrow{r} C^*(G|_F) \to 0$$

where $i : C_c(G|_U) \to C_c(G)$ is the extension of functions, and $r : C_c(G) \to C_c(G|_F)$ is the restriction of functions.

The second is a groupoid analogue of tensoring C^* algebras.

Theorem 6.16 *[ADR00] Let G_1, G_2 be locally compact groupoids equipped with Haar systems, and suppose that G_1 is amenable. $C^*(G_1) = C_r^*(G_1)$ is nuclear. $G_1 \times G_2$ is locally compact and*

$$C^*(G_1 \times G_2) \simeq C^*(G_1) \otimes C^*(G_2) \text{ and } C_r^*(G_1 \times G_2) \simeq C^*(G_1) \otimes C_r^*(G_2)$$

Lastly, we will need the six exact sequence KK-theory.

Theorem 6.17 *(Six terms exact sequence) Given SES $0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$ of C^* algebras Q another C^* algebra. With more assumptions, i.e. all C^* algebras are nuclear, there is six term exact sequence (and similarly in other variable)*

$$\begin{array}{ccccc} KK(Q,I) & \xrightarrow{i_*} & KK(Q,A) & \xrightarrow{p_*} & KK(Q,B) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\delta} \\ KK_1(Q,B) & \xleftarrow[p_*]{} & KK_1(Q,A) & \xleftarrow[i_*]{} & KK_1(Q,I) \end{array}$$

From now on we will heavily use these three properties, the Kasparov axioms, and Prop. 6.10. Consider deformation groupoid.

$$G = G_1 \times \{0\} \cup G_2 \times (0,1] \rightrightarrows G^{(0)} = M \times [0,1]$$

Let $ev_1 : C^*(G) \to C^*(G_2)$ be evaluation at 1, $[ev_1] \in KK(C^*(G), C^*(G_2))$, as Prop. 6.5.

Lemma 6.18 *There exists* $[ev_0]^{-1} \in KK(C^*(G_1), C^*(G))$ *such that*

$$[ev_0] \otimes [ev_0]^{-1} = 1_{C^*(G)}, \quad [ev_0]^{-1} \otimes [ev_0] = 1_{C^*(G_1)}$$

Proof: We apply Thm. 6.15 to obtain

$$C^*(G|_{M\times(0,1]}) \simeq C^*(G_2) \otimes C_0((0,1])$$

$$C^*(G|_{M\times\{0\}}) \simeq C^*(G_1)$$

Combining with Thm. 6.16, we have the exact sequence of C^* algebras

$$0 \to C^*(G_2) \otimes C_0((0,1]) \xrightarrow{i_{M\times(0,1]}} C^*(G) \xrightarrow{ev_0} C^*(G_1) \to 0$$

where $i_{M\times(0,1]}$, ev_0 are inclusion and evaluation maps respectively.

The C^* algebra $C^*(G_2) \otimes C_0((0,1])$. By functorial compatibility of the Kasparov product and Thm. 6.17,

$$(ev_0)_* : KK(A, C^*(G)) \to KK(A, C^*(G_1)), \quad x \mapsto x \otimes [ev_0]$$

Let $A = C^*(G_1)$ to obtain $[ev_0]^{-1}$. □

Definition 6.19 The KK*-element associated to the deformation groupoid* G is defined by

$$\delta = [ev_0]^{-1} \otimes [ev_1] \in KK(C^*(G_1), C^*(G_2))$$

We now formulate the index maps in KK-theory and groupoids.

Set $G_1 = TM$, $G_2 = M \times M$ in Def. 6.19, we obtain an element,

$$\partial_M \in KK(C_0(T^*M), \mathcal{K}) \simeq KK(C_0(T^*M), \mathbb{C}).$$

Definition 6.20

$$\mathrm{Ind}_a = \partial_M$$

$$\begin{array}{ccc} KK(\mathbb{C}, C_0(T^*M)) & \longrightarrow & KK(\mathbb{C}, \mathcal{K})) \\ \simeq\uparrow & & \simeq\uparrow \\ K_0(C_0(T^*M)) & \longrightarrow & \mathbb{Z} \end{array}$$

$$x \mapsto x \otimes \partial_M$$

That this map is analytic index map is done by G-operators, [DL08, Pg. 54].

The Topological Index Map

We break down Def. 4.13. Given tubular neighborhood embedding $i : M \hookrightarrow \mathbb{R}^n$, $N \to M$ its normal bundle, and $T^*N \to T^*M$ associated complex bundle. The Thom isomoprhism

$$K_0(C^*(TM)) \simeq K(T^*M) \xrightarrow{\simeq Thom} K(T^*N) \simeq K_0(C^*(TN))$$

is given[3] by a Thom element, [Kas80]

$$T \in KK(C^*(TM), C^*(TN))$$

Hence, the morphism is $x \mapsto x \otimes T$.

Definition 6.21 Let $\pi : E \to X$ be a vector bundle. We have the groupoids,

$$\pi_{[0,1]} : E \times [0,1] \to X \times [0,1]$$
$$G_X^t \rightrightarrows X \times [0,1]$$
$$\mathcal{G}_E^t \rightrightarrows E \times [0,1]$$
$$\mathcal{T}_E^t := \mathcal{G}_E^t \times \{0\} \sqcup \pi_{[0,1]}^*(\mathcal{G}_X^t) \times (0,1] \rightrightarrows E \times [0,1] \times [0,1]$$

and by restriction of $\mathcal{T}_E^t$ to $E \times \{0\} \times [0,1]$ we obtain the *Thom groupoid*, $\mathcal{T}_E$.

We apply this to the special case of the normal bundle $N \to M$ to obtain

Definition 6.22

$$\mathcal{T}_N = TN \times \{0\} \cup p^*(TM) \times (0,1] \rightrightarrows N \times [0,1]$$

where $p^*(TM)$ is the pullback groupoid along $p : N \to M$ of $TM \rightrightarrows M$

Set $G_1 = TN$, $G_2 = p^*TM$, in Def. 6.19, we obtain element

$$\tau_N \in KK(C^*(TN), C^*(p^*TM)) \simeq KK(C^*(TN), C^*(TM))$$

via Morita equivalence. But this is exactly the inverse of T, [DLN06],

Theorem 6.23 *$\tau_N = T^{-1}$, where τ_N given in Def. 6.22, T being the KK-equivalence inducing Thom isomoprhism.*

From the embedding $* \hookrightarrow \mathbb{R}^n$, with the normal bundle $\mathbb{R}^n \to *$, $\tau_{\mathbb{R}^n} \in KK(C^*(T\mathbb{R}^n), \mathbb{C})$ coincides with the isomoprhism induced by Bott element by Thm. 6.23. This is thus our topological index

Definition 6.24

$$\mathrm{Ind}_t = \tau_{\mathbb{R}^n} \circ i_* \circ \tau_N^{-1}$$

$$\begin{array}{ccccc}
KK(\mathbb{C}, C^*(TM)) & \longrightarrow & KK(\mathbb{C}, C^*(TN)) & \longrightarrow & KK(\mathbb{C}, C^*(TN)) \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
K(C^*(TM)) & \longrightarrow & K(C^*(TN)) & \longrightarrow & K(C^*T\mathbb{R}^n) \simeq \mathbb{Z}
\end{array}$$

[3] Given $x \in KK(A,B)$ the Kasparov product induces maps $KK(,A) \simeq K_0(A) \to K_0(B) \simeq KK(\mathbb{C}, B), \alpha \otimes \alpha x$. In many cases, this map is surjective.

6.3 Proof of the Atiyah-Singer Index Theorem

We construct the Thom groupoid at a higher lever. Define $p \times id : N \times [0,1] \to M \times [0,1]$, and pullback over $\mathcal{G}^t_M$, giving

Definition 6.25

$$\tilde{\mathcal{T}}_N = \mathcal{G}^t_N \times \{0\} \sqcup (p \times id)^*(\mathcal{G}^t_M) \times (0,1] \rightrightarrows N \times [0,1] \times [0,1]$$

By Def. 6.19, we have an element $\tilde{\tau}_N \in KK(C^*(\mathcal{G}^t_N), C^*(\mathcal{G}^t_M))$.

Theorem 6.26 *The following diagram commutes.*

$$\begin{array}{ccccc}
\mathbb{Z} & = & \mathbb{Z} & = & \mathbb{Z}\\
e_1^M \uparrow & (Q1) & e_1^N \uparrow & & e_1^{\mathbb{R}^n} \uparrow\\
K_0(C^*(\mathcal{G}^t_M)) & \xleftarrow[(-)\otimes\tilde{\tau}_N]{} & K_0(C^*(\mathcal{G}^t_N)) & \xrightarrow{\tilde{i}_*} & K_0(C^*(\mathcal{G}^t_{\mathbb{R}^n}))\\
e_0^M \downarrow \simeq & & e_0^N \downarrow \simeq & & e_0^{\mathbb{R}^n} \downarrow \simeq\\
K_0(C^*(TM)) & \xleftarrow[(-)\otimes\tau_N]{\simeq} & K_0(C^*(TN)) & \xrightarrow{i_*} & K_0(C^*(T\mathbb{R}^n))
\end{array}$$

Proof: We show (Q1). Others are similar, using essentially:

- functoriality of pairing with explicit restrictions or inclusions as maps.
- inverses via that described in Def. 6.19
- explicit Morita equivalences, via that induced from pull backs

(Q1) is broken to two pieces. Let $p : N \to M$ denote the normal bundle, inducing maps

$$\mathcal{G}^t_N \to \mathcal{G}^t_M, \quad (p \times id)^*\mathcal{G}^t_M \to \mathcal{G}^t_M, \quad N \times N \to M \times M$$

$$\begin{array}{ccc}
\mathcal{G}^t_N & \xrightarrow{p} & \mathcal{G}^t_M\\
\downarrow & & \downarrow\\
N \times N & \longrightarrow & M \times M
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
KK(\mathbb{C}, C^*(\mathcal{G}^t_N)) & \xrightarrow{p_*} & KK(\mathbb{C}, C^*(\mathcal{G}^t_M))\\
\downarrow & & \downarrow\\
KK(\mathbb{C}, C^*(N \times N) & \longrightarrow & KK(\mathbb{C}, C^*(M \times M)\\
\simeq \uparrow & & \simeq \uparrow\\
\mathbb{Z} & = & \mathbb{Z}
\end{array}$$

Now note that $ev_0 j_0 = id$, where j_0 is the zero section inclusion. By commutativity of the left diagram, the induced maps are in fact that defined by as 6.19. This implies, $p = \cdot \otimes \tilde{\tau}_N$,

$$
\begin{array}{ccc}
\mathcal{G}_N^t & \xrightarrow{p} & \mathcal{G}_M^t \\
{\scriptstyle ev_0}\uparrow\ \downarrow{\scriptstyle j_0} & & \uparrow \\
\tilde{T}_N & \xrightarrow{ev_1} & p^*\mathcal{G}_M^t
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
KK(\mathbb{C}, C^*(\mathcal{G}_N^t)) & \xrightarrow{(-)\otimes\tilde{\tau}_N = p_*} & KK(\mathbb{C}, C^*(\mathcal{G}_M) \\
\downarrow{\scriptstyle (-)\otimes[ev_0]^{-1}} & & \uparrow \\
KK(\mathbb{C}, C^*(\mathcal{T}_N)) & \xrightarrow{(-)\otimes[ev_1]} & KK(\mathbb{C}, C^*((p \times id)^*\mathcal{G}_M^t))
\end{array}
$$

where the map $p^*\mathcal{G}_M^t \to \mathcal{G}_M^t$ induces the Morita equivalence map. □

The outer maps are equal, which states the Atiyah Singer Index Theorem.

Theorem 6.27

$$\mathrm{Ind}_a = \mathrm{Ind}_t$$

Epilogue

There are many exciting directions to explore. In continuation, there is the work by Alain Connes, [Con95], and the Baum-Connes conjecture, [Sch16]. In the algebraic side, there is the higher index theorem, [ENN96], [Nis97], that looks at the Chern Character from algebraic K-theory to periodic cyclic homology. In the mathematical physics sides, there is the Heat kernel proof, [Get83], discussed in detail in [IN13].

Lastly, many computational aspects were left out. Wealth of material is in Michelson and Lawson's [HBL90], which includes applications to Kähler geometry and divisibility theorems for characteristic numbers.

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